# ORDINAL SUMS OF PROJECTIVES IN VARIETIES OF LATTICES

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Projective lattices were characterized in [4] by four conditions; see also [3]. The fourth condition was the following.

> For each $a \in L$ there are two finite sets $\mathrm{A}(a) \subseteq \{x \in L : x \geq a\}$ and $\mathrm{B}(a) \subseteq \{x \in L : x \leq a\}$ such that if $a \leq b$ then $\mathrm{A}(a) \cap \mathrm{B}(b) \neq \emptyset$.

(A for 'above'; B for 'below.') A lattice satisfying this condition is called *finitely separable*. It is easy to see that every countable lattice is finitely separable. We used this to show the following surprising result: *the ordinal sum of two free lattices is projective if and only if one of them is finitely generated or both are countable.*

In this note we give a complete characterization of when the ordinal sum of two lattices (the lattice obtained by placing the second lattice on top of the first) is projective. This characterization applies not only to the class of all lattices, but to any variety of lattices. In particular, to the class of distributive lattices.

## 1. An interpolation result for relatively free lattices

In this section we prove an interpolation result of independent interest. Let $\mathcal{V}$ be a variety of lattices and let $\mathbf{F}_{\mathcal{V}}(X)$ denote the free $\mathcal{V}$–lattice generated by $X$. The following lemma is elementary.

**Lemma 1.** *Let $Z$ be a finite set and let $Y$ be a subset. Then every element of $\mathbf{F}_{\mathcal{V}}(Z)$ is either below $\bigvee Y$ or above $\bigwedge(Z - Y)$.*

*Proof.* If $\mathcal{V}$ is the trivial variety then the result is clear. Otherwise the result follows from examining the homomorphism from $\mathbf{F}_{\mathcal{V}}(X)$ to $\mathbf{2}$ which maps the elements of $Y$ to 0 and the other generators to 1. □

For $w \in \mathbf{F}_{\mathcal{V}}(X)$ define the *rank* of $w$, denote $r(w)$, to be the the least integer $k$ such that there is a term of length $k$ representing $w$. For $Y$ a finite

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subset of $X$, let $\tau_Y$ be the endomorphism of $\mathbf{F}_{\mathcal{V}}(X)$ extending the map on $X$:

$$\tau_Y(z) = \begin{cases} z & \text{if } z \in Y \\ \bigvee Y & \text{if } z \notin Y \end{cases}$$

**Lemma 2.** *Let $\mathcal{V}$ be a nontrivial variety of lattices, let $Y$ be a finite subset of $X$, and let $\tau = \tau_Y$. Suppose $w \le \bigvee Y$ in $\mathbf{F}_{\mathcal{V}}(X)$. Then $w \le \tau(w)$ and the $r(\tau(w)) \le r(w)$. Moreover, if $U$ is a subset of $X$ such that $w$ lies in the sublattice generated by $U$, then $\tau(w)$ also lies in this sublattice.*

*Proof.* We induct on the rank of $w$. If $w$ is in $X$, then $w \le \bigvee Y$ implies $w \in Y$ since $\mathcal{V}$ is nontrivial; so the result is clear. Let $Z$ be a finite subset of $X$ containing $Y$ such that $w$ lies in the sublattice generated by $Z$. If $w = w_1 \vee \cdots \vee w_n$ where the rank of each $w_i$ is less than the rank of $w$, then an easy induction argument applies.

So assume $w = w_1 \wedge \cdots \wedge w_n$ where the rank of each $w_i$ is less than the rank of $w$. Since $\bigvee Y$ is meet prime by Lemma 1, there is an $i$ with $w_i \le \bigvee Y$. By renumbering we may assume that $w_i \le \bigvee Y$ for $1 \le i \le m$ and, by Lemma 1, $w_j \ge \bigwedge(Z - Y)$ for $j > m$. By induction, $w_i \le \tau(w_i)$ and $r(\tau(w_i)) \le r(w_i)$, for $i \le m$. For $j > m$, $\tau(w_j) \ge \tau(\bigwedge(Z - Y)) = \bigvee Y$. Clearly the image of $\tau$ lies below $\bigvee Y$; hence $\tau(w_i) \le \tau(w_j)$ if $i \le m$ and $j > m$. Thus

$$\tau(w) = \bigwedge_{i=1}^{n} \tau(w_i) = \bigwedge_{i=1}^{m} \tau(w_i) \ge \bigwedge_{i=1}^{m} w_i \ge w$$

and $r(\tau(w)) \le r(w)$. The final statement of the lemma is also clear. □

Now suppose that $w \le u$ in $\mathbf{F}_{\mathcal{V}}(X)$. Let $Y$ and $Z$ be a finite subsets of $X$ such that the lattice generated by $Y$ contains $u$ and the lattice generated by $Z$ contains $w$. Let $\tau = \tau_Y$. Since $u \le \bigvee Y$, Lemma 2 gives

$$w \le \tau(w) \le \tau(u) = u \tag{1}$$

and $\tau(w)$ lies in the sublattice generated by $Y \cap Z$. In the case $w = u$, we see that *if $w$ lies in the sublattice generated by $Y$ and by $Z$, then it lies in the sublattice generated by $Y \cap Z$.* Thus, for each $w \in \mathbf{F}_{\mathcal{V}}(X)$ there is a unique smallest subset $Y$ of $X$ such that $w$ is in the sublattice generated by $Y$. We let $\operatorname{var}(w)$ denote this subset.

Now for our interpolation theorem.

**Theorem 3.** *If $w \le u$ in $\mathbf{F}_{\mathcal{V}}(X)$, there is a $v$ with $w \le v \le u$ such that $\operatorname{var}(v) \subseteq \operatorname{var}(w) \cap \operatorname{var}(u)$ and $\mathrm{r}(v) \le \min(\mathrm{r}(w), \mathrm{r}(u))$.*

*Proof.* If $\mathrm{r}(w) \le \mathrm{r}(u)$, let $v = \tau_{\operatorname{var}(u)}(w)$ and the result follows from (1) and Lemma 2. If $\mathrm{r}(w) > \mathrm{r}(u)$, then a dual argument implies the result. □

Using this interpolation theorem we can show that every lattice projective in $\mathcal{V}$ is finitely separable.

**Theorem 4.** *If $\mathbf{L}$ is a projective lattice in $\mathcal{V}$ then $\mathbf{L}$ is finitely separable.*

*Proof.* Finite separability is clearly preserved by retractions; thus it suffices to prove this theorem for $\mathbf{L} = \mathbf{F}_{\mathcal{V}}(X)$. For $a \in L$, let

$$\mathrm{A}(a) = \{w \in \mathbf{F}_{\mathcal{V}}(X) : w \geq a, \ \mathrm{var}(w) \subseteq \mathrm{var}(a), \text{ and } \mathrm{r}(w) \leq \mathrm{r}(a)\}$$

and define $\mathrm{B}(a)$ dually. The theorem follows from Theorem 3. □

## 2. Projective ordinal sums

In this section we characterize when the ordinal sum of two lattices is projective in $\mathcal{V}$. We begin with some results of independent interest.

**Theorem 5.** *If* $\mathbf{P}$ *is a partially ordered set, then* $\mathbf{F}_{\mathcal{V}}(\mathbf{P})$ (*the* $\mathcal{V}$ *lattice freely generated by* $P$ *subject to the order relations of* $\mathbf{P}$) *is projective if and only if* $\mathbf{P}$ *is finitely separable.*

*Proof.* The proof of Lemma 7 in [4] also proves this result. □

**Corollary 6.** *If the ordinal sum* $\mathbf{L} = \mathbf{L}_0 \dotplus \mathbf{L}_1$ *is projective in* $\mathcal{V}$, *then both* $\mathbf{L}_0$ *and* $\mathbf{L}_1$ *are projective in* $\mathcal{V}$.

*Proof.* Clearly $\mathbf{L}_0 \dotplus \mathbf{1}$ is a retraction of $\mathbf{L}$ and hence projective. Let $X_0$ be a set with the same cardinality as $\mathbf{L}_0$. Then, by Theorem 5, $\mathbf{F}_{\mathcal{V}}(X_0) \dotplus \mathbf{1}$ is projective. Thus if $f : \mathbf{F}_{\mathcal{V}}(X_0) \dotplus \mathbf{1} \twoheadrightarrow \mathbf{L}_0 \dotplus \mathbf{1}$ is a homomorphism extending a map from $X_0$ onto $L_0$, then there is a homomorphism $g : \mathbf{L}_0 \dotplus 1 \rightarrowtail \mathbf{F}_{\mathcal{V}}(X_0)$ such that $fg$ is the identity on $\mathbf{L}_0 \dotplus \mathbf{1}$ and $gf$ is a retraction of $\mathbf{F}_{\mathcal{V}}(X_0) \dotplus \mathbf{1}$ onto $\mathbf{L}_0 \dotplus \mathbf{1}$. Now $gf \restriction \mathbf{F}_{\mathcal{V}}(X_0)$ is clearly a retraction of $\mathbf{F}_{\mathcal{V}}(X_0)$ onto $\mathbf{L}_0$, showing that $\mathbf{L}_0$ is projective in $\mathcal{V}$. □

**Theorem 7.** *Let* $\mathcal{V}$ *be a variety of lattices and let* $\mathbf{L} = \mathbf{L}_0 \dotplus \mathbf{L}_1$, *where* $\mathbf{L}_i \in \mathcal{V}$ *for* $i = 0, 1$. *Then* $\mathbf{L}$ *is projective in* $\mathcal{V}$ *if and only if both* $\mathbf{L}_0$ *and* $\mathbf{L}_1$ *are and one of the following hold:*

1. $\mathbf{L}_0$ *has a greatest element.*
2. $\mathbf{L}_1$ *has a least element.*
3. $\mathbf{L}_0$ *has a countable cofinal chain and* $\mathbf{L}_1$ *has a countable coinitial chain.*

*Proof.* We begin with a lemma.

**Lemma 8.** *If* $\mathbf{L}_0$ *and* $\mathbf{L}_1$ *are finitely separable and 1, 2, or 3 holds, then* $\mathbf{L}_0 \dotplus \mathbf{L}_1$ *is finitely separable.*

*Proof.* Assume that 3 holds; the other two cases are easier. Let $c_0 < c_1 < c_2 < \cdots$ be a cofinal chain in $\mathbf{L}_0$ and let $d_0 > d_1 > d_2 > \cdots$ be a coinitial chain in $\mathbf{L}_1$. By hypothesis there are function $\mathrm{A}_{\mathbf{L}_i}$ and $\mathrm{B}_{\mathbf{L}_i}$ which witness the finite separability of $\mathbf{L}_i$, $i = 0$ and 1.. Define

$$\mathrm{A}'(c_i) = \{d_0, d_1, \ldots, d_i\} \cup \{c_i\} \quad \text{and} \quad \mathrm{B}'(d_i) = \{c_0, c_1, \ldots, c_i\} \cup \{d_i\}.$$

Let $a \in L_0$ and suppose that $a \leq c_i$ but $a \nleq c_j$ if $j < i$. Set $\mathrm{A}'(a) = \mathrm{A}'(c_i)$ and define

$$\mathrm{A}_{\mathbf{L}}(a) = \mathrm{A}_{\mathbf{L}_0}(a) \cup \mathrm{A}'(a)$$

We define $\mathrm{B}_{\mathbf{L}}(b)$, for $b \in \mathbf{L}_1$, by dual considerations. Of course, $\mathrm{B}_{\mathbf{L}}(a) = \mathrm{B}_{\mathbf{L}_0}(a)$ and $\mathrm{A}_{\mathbf{L}}(b) = \mathrm{A}_{\mathbf{L}_0}(b)$. With these definitions, it is easy to verify that $\mathbf{L}$ is finitely separable. □

Suppose that $\mathbf{L}_0$ and $\mathbf{L}_1$ are projective in $\mathcal{V}$ and that one of 1, 2, or 3 holds. Let $\mathbf{P}_i$ be $\mathbf{L}_i$ as a partially ordered set, and let $\mathbf{P} = \mathbf{P}_0 \dotplus \mathbf{P}_1$. By Lemma 8, $\mathbf{L}$ is finitely separable, and so by Theorem 5, $\mathbf{F}_{\mathcal{V}}(\mathbf{P}_0) \dotplus \mathbf{F}_{\mathcal{V}}(\mathbf{P}_1)$ is projective. Since $\mathbf{L}_i$ is projective, it is a retract of $\mathbf{F}_{\mathcal{V}}(\mathbf{P}_i)$, for $i = 1, 2$. Clearly these retractions can be patched together to show that $\mathbf{L}_0 \dotplus \mathbf{L}_1$ is a retract of $\mathbf{F}_{\mathcal{V}}(\mathbf{P}_0) \dotplus \mathbf{F}_{\mathcal{V}}(\mathbf{P}_1)$. Thus $\mathbf{L}$ is projective in $\mathcal{V}$.

This proves one direction of he theorem. For the other, suppose that $\mathbf{L}$ is projective in $\mathcal{V}$. By Corollary 6, both $\mathbf{L}_0$ and $\mathbf{L}_1$ are projective. Since $\mathbf{L}$ is projective, functions A and B exist witnessing its finite separability. Let $\mathrm{A}_{\mathbf{L}_i}(x) = \mathrm{A}(x) \cap L_i$ and define $\mathrm{B}_{\mathbf{L}_i}$ similarly.

Suppose that 1, 2, and 3 all fail. Since 3 fails, we may assume by duality that $\mathbf{L}_0$ has no cofinal chain. Inductively we construct a sequence $d_0 > d_1 > d_2 > \cdots$ in $\mathbf{L}_1$ such that

$$d_n \notin \mathrm{OrdFil}\,[\mathrm{B}_{\mathbf{L}_1}(d_m)] \qquad \text{if } n > m. \tag{2}$$

That this can be done follows from the fact that, for any $y \in L_1$, $L_1 - \mathrm{OrdFil}\ \mathrm{B}_{\mathbf{L}_1}(y)$ must be infinite since otherwise the meet of these elements and of $\mathrm{B}_{\mathbf{L}_1}(y)$ would be a least element of $\mathbf{L}_1$, contrary to our assumption that 2 fails. (2) immediately implies that

$$\mathrm{B}_{\mathbf{L}_1}(d_i) \cap \mathrm{B}_{\mathbf{L}_1}(d_j) = \emptyset \qquad \text{if } i \neq j. \tag{3}$$

If $\mathrm{OrdIdl}\,[\bigcup_{i=0}^{\infty} \mathrm{B}_{\mathbf{L}_0}(d_i)] = L_0$, then $\mathbf{L}_0$ would be countable and so have a countable cofinal chain, contrary to our assumption. Thus let $c \notin \mathrm{OrdIdl}(\bigcup_{i=0}^{\infty} \mathrm{B}_{\mathbf{L}_0}(d_i))$. Of course $c < d_i$ and hence there is an $e_i \in L$ with

$$e_i \in \mathrm{A}(c) \cap \mathrm{B}_{\mathbf{L}_1}(d_i).$$

By (3), these $e_i$'s must be distinct. But this forces $\mathrm{A}(c)$ to be infinite. □

As an application we show how Theorem 7 implies a theorem of Balbes [1] on projective distributive lattices.

**Theorem 9.** (Balbes [1]) *Let $\mathbf{L}$ is a projective distributive lattice generated by $A \cup B$ where $A$ and $B$ are antichains, each element of $A$ is join irreducible, each element of $B$ is meet irreducible, and $a \leq b$ for each $a \in A$ and $b \in B$. Then either $A \cup B$ is countable or $A$ or $B$ is finite.*

*Proof.* Let $\mathbf{L}_0$ be the sublattice generated by $A$ and $\mathbf{L}_1$ be the sublattice generated by $B$. Since every element of a projective distributive lattice is both a join of join irreducible elements and a meet of meet irreducible elements (see [2] or [4]), $\mathbf{L} = \mathbf{L}_0 \dotplus \mathbf{L}_1$. By Theorem 7, one of the conditions 1, 2, or 3 must hold.

If $\mathbf{L}_0$ has a greatest element $u$ then, since $\mathbf{L}_0$ is generated by $A$, $u = \bigvee A'$ for some finite subset $A'$ of $A$. But in any distributive lattice join irreducible elements are join prime which implies $A = A'$ and so is finite. Thus by duality we may assume that $\mathbf{L}_0$ has a countable cofinal chain and $\mathbf{L}_1$ has a countable coinitial chain. Since $\mathbf{L}_0$ is generated by $A$, each element $u_i$ of the cofinal chain lies in the sublattice generated by a finite subset $A_i \subseteq A$. Let $a \in A$. Then $a \leq u_i$ for some $u_i$ in the cofinal chain. Then $a \leq u_i \leq \bigvee A$. Since $A$ is an antichain, this implies that $a \in A_i$. Thus $A$ is the union of the $A_i$'s and so is countable. Similarly, $B$ is countable. □

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