

Singular Covers in Free Lattices

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Abstract. A covering $a \succ b$ in a lattice is called a singular cover if a is join irreducible and b is meet irreducible. A classification of the singular covers which occur in free lattices is given.

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1. Introduction

If a covers b (denoted $a \succ b$) in a lattice L , we call this a *singular cover* (or the pair (a, b) a *singular covering pair*) if a is join irreducible and b is meet irreducible. This note characterizes such covers in free lattices. Whenever $a \prec b$ in a lattice L then, by Dilworth's characterization of lattice congruences, there is a unique largest congruence $\psi(a, b)$ not containing (a, b) . Moreover in this situation $L/\psi(a, b)$ is subdirectly irreducible. When L is a free lattice, these subdirectly irreducible lattices are always finite and are called splitting lattices. Such lattices were extensively studied by McKenzie [13]. Notice that if (a, b) is a singular covering pair, then the congruence generated by (a, b) , $\theta(a, b)$, identifies only a and b . Thus if a subdirectly irreducible lattice has a singular covering pair (a, b) then this is the only critical quotient. (A *critical quotient* in a subdirectly irreducible lattice is one that generates the unique minimal nontrivial congruence.) Many important splitting lattices have singular covers for their critical quotients. For example, the congruence lattices of the finitely generated free algebras in Polin's variety have this property [4]. Another such class of lattices is investigated in [3].

If $L = FL(X)/\psi(w, v)$, where $w \succ v$, is a splitting lattice with a singular cover, it does not follow that (w, v) is a singular covering in $FL(X)$. However, the question of what are the singular coverings in $FL(X)$ is still interesting. In their study of covers in free lattices, Freese and Nation needed to understand singular covers in free lattices which satisfied some additional assumptions. Those authors were able to show that if $w \succ v$ is a singular cover satisfying these additional hypotheses, then either v is the join of two

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atoms or w is the meet of two coatoms. They asked if there were other singular covers in free lattices. Ježek and Slavík were able to prove for any singular cover in a free lattice there is a generator comparable to both the elements of the pair. Using this they were able to prove that the only singular covers in $FL(3)$ were those described above. Freese and Nation, using the results of [9] and [6], were then able to extend this result to all free lattices.

THEOREM. *If $w \succ v$ is a singular cover in a free lattice, then either v is the join of two distinct atoms or w is the meet of two distinct coatoms.*

In $FL(3)$ $x(xy + xz + yz) \succ xy + xz$ is indeed a singular covering pair. Likewise, if we let a_i denote the atom $\prod_{j \neq i} x_j$ of $FL(n)$, ($n \geq 4$), then for each i, j with $1 \leq i < j \leq n$,

$$\prod_{k \neq i, j} (a_i + a_j + a_k) \succ a_i + a_j$$

is a singular covering pair in $FL(n)$. Of course, the duals of these are also singular covering pairs.

Our proof of the theorem will use quite a bit of the machinery developed in [9] and [6] for studying covers in free lattices. However, the argument for $FL(3)$ involves some interesting and different techniques, so we will sketch some of the important ideas here.

In [15] Whitman proved the free lattices satisfy the following continuity property. We write $a_n \uparrow a$ if $\{a_n : n \geq 0\}$ is an ascending chain and a is the least upper bound for this chain. Whitman showed that if $a_n \uparrow a$ in $FL(X)$ and $u \in FL(X)$ then $a_n \cdot u \uparrow a \cdot u$. From this it follows that if $a_n \uparrow a$ and $b_n \downarrow b$, $b \leq a$, and $b_m \not\leq a_n$ for all m and n , then the interval $[b, a]$ contains no singular cover. Indeed, if $w \succ v$ were a singular cover in $[b, a]$, then $a_n \cdot w \uparrow a \cdot w = w$. If $a_n \cdot w < w$ then $a_n \cdot w \leq v$. Hence, for some n , $a_n \cdot w = w$, i.e., $w \leq a_n$. By duality $b_m \leq v$ for some m . Hence, $b_m \leq a_n$, a contradiction.

Let a be the meet of the coatoms and b the join of the atoms in $FL(X)$. Let $X = \{x_1, \dots, x_n\}$ and for $c \in FL(X)$ define

$$f(c) = \prod_{i < j} (c \cdot x_i + c \cdot x_j).$$

We define $g(c)$ dually. Let $b_0 = a$ and let $b_{n+1} = f(b_n)$ and define a_n dually. Then, with some work, one can show that $a, b, a_n, b_n, n \geq 0$, satisfy the hypothesis of the result of the previous paragraph. Hence $[b, a]$ contains no singular cover. Notice that $[b, a]$ is precisely the set of all elements which are not comparable with a generator. With the aid of this and some additional work one can show that the only singular covers in $FL(3)$ are those described in the theorem.

2. Preliminaries

Before we begin the proof of the theorem we will need to review some basic facts about free lattices and their covers. In [10] Whitman solved the word problem for free lattices. We assume the reader is familiar with his solution. Part of his solution is the condition (W):

$$(W) \quad \begin{array}{l} v_1 \dots v_n = v \leq u = u_1 + \dots + u_m \text{ implies} \\ v_i \leq u \text{ for some } i \text{ or } v \leq u_j \text{ for some } j. \end{array}$$

which holds in $FL(X)$. Note this implies that every element of $FL(X)$ is either meet or join irreducible. We say that $\{w_1, \dots, w_k\}$ *refines* $\{u_1, \dots, u_m\}$, and we write $\{w_1, \dots, w_k\} \ll \{u_1, \dots, u_m\}$, if for each i there is a j such that $w_i \leq u_j$. Whitman also showed that for each $w \in FL(X)$ there is a term of minimal length representing w , unique up to commutativity and associativity. If $w = w_1 + \dots + w_k$ is this representation, where each w_i is not formally a join, and if $w = u_1 + \dots + u_m$, then $\{w_1, \dots, w_k\} \ll \{u_1, \dots, u_m\}$. We call $\{w_1, \dots, w_k\}$ the *canonical joinands* of w . It follows from this that if $a < b$ in $FL(X)$ then there is a unique largest element c with $c \geq a$ but $c \not\geq b$. Namely, c is the unique canonical meetand of a which is not above b . In particular, if w is completely join irreducible, i.e., w is join irreducible and has a unique lower cover (always denoted w_*), then there is a unique largest element, denoted $\kappa(w)$, satisfying $\kappa(w) \geq w_*$ and $\kappa(w) \not\geq w$, and $\kappa(w)$ is the unique canonical meetand of w_* which is not above w . If q is completely meet irreducible, we let $\kappa'(q)$ be the map dual to κ . Note κ and κ' are inverses of each other.

Another consequence of this refinement property of canonical joins is the fact that $FL(X)$ satisfies Jónsson's semidistributive laws, (SD_{\vee}) and (SD_{\wedge}) :

$$(SD_{\vee}) \quad a + b = a + c \text{ implies } a + b = a + b \cdot c.$$

If w is a join, say $w = \Sigma w_i$, canonically, then the following lemma shows that the lower covers of w are determined by the completely join irreducible canonical joinands of w .

LEMMA 1.1 (Corollary 2.4 [9]). *Let $w = \Sigma w_i$ canonically in $FL(X)$. If w_i is completely join irreducible then $w \succ w \cdot \kappa(w_i)$. Conversely, if $w \succ v$ and w_i is the unique joinand not below v , then w_i is completely join irreducible and $w_{i*} = v \cdot w_i$. Moreover in this case every element below w is either below v or above w_i .*

We will need several technical theorems about covers in free lattices from [9]. If w has the form

$$w = \prod_i \left(\sum_j w_{ij} \right) \prod_k x_k \quad (1)$$

where the x_k 's are generators, then $J(w)$ is defined recursively to be $\{w\}$ if w is a meet of variables, and

$$\{w\} \cup \bigcup_{i,j} J(w_{ij})$$

otherwise. The set $M(w)$, for w meet irreducible, is defined dually.

THEOREM 1.2 (Theorem 4.3 [9]). *Let w be a join irreducible element of $FL(X)$, X finite, and let*

$$w_{\dagger} = \sum \{u \in J(w) : u < w\} \text{ and } K(w) = \{v \in J(w) : w_{\dagger} + v \not\geq w\}.$$

Then w is completely join irreducible if and only if

- (i) every $u \in J(w) - \{w\}$ is completely join irreducible, and
- (ii) $w \not\leq \Sigma K(w)$.

Notice that in particular if w is completely join irreducible then so is every element of $J(w)$.

THEOREM 1.3 (Theorem 4.4 [9]). *If w is a completely join irreducible element of $FL(X)$ and has the canonical form (1) where each $x_k \in X$, then for each i there is exactly one j with $w_{ij} \not\leq w$.*

By rearranging elements we can always assume that $j = 1$, so that we always have $w_{i1} \not\leq w$.

THEOREM 1.4 (Theorem 4.7 [9]). *If w is completely join irreducible then the canonical meetands of w_* are $\{\kappa(w)\} \cup \{w_i : w_i \not\geq \nu(w)\}$.*

The next theorem gives a useful characterization of three element intervals in free lattices.

THEOREM 1.5 (Theorem 10.2 [9]). *Suppose w is join irreducible and the middle element of a three element interval $w_* < w < u$. Then $w \notin X$, u is join reducible, and if q is the canonical meetand of w not above u then $w = q \cdot u$ canonically and q is completely meet irreducible.*

If we define an equivalence relation \sim by $a \sim b$ if there is a sequence $a = a_0, a_1, \dots, a_n = b$ such a_i covers or is covered by a_{i+1} , then the connected components (of the covering relation) of the free lattices are the equivalence classes of this relation, thought of as subpartially ordered sets of the free lattice. A complete description of all these connected components is given in [6]. We present those parts of this description that we require below.

For $n \geq 4$ the connected component of the least element, $0 = \Pi x_i$, of $FL(n)$ consists of 0 , the n atoms, their $\binom{n}{2}$ pairwise joins, and $\binom{n}{2}$ elements covering each of these pairwise joins. This component, for $n = 4$, is diagrammed in Figure 1. The connected component for $n = 3$ is also given by the diagram below. In both diagrams the singular covers are labelled. In the first, $a_i = \Pi_{j \neq i} x_j$ is an atom.

Note that the join of any two atoms of a free lattice is the lower element of a singular pair.

3. The Proof

Now we are ready to begin the proof of the theorem. We begin with two lemmas. From now on we shall assume that $w > v$ is a singular cover in $FL(X)$ and that the canonical form of w is given by (1), where, as before, $x_k \in X$ and for each i , $w_{i1} \not\leq v$.

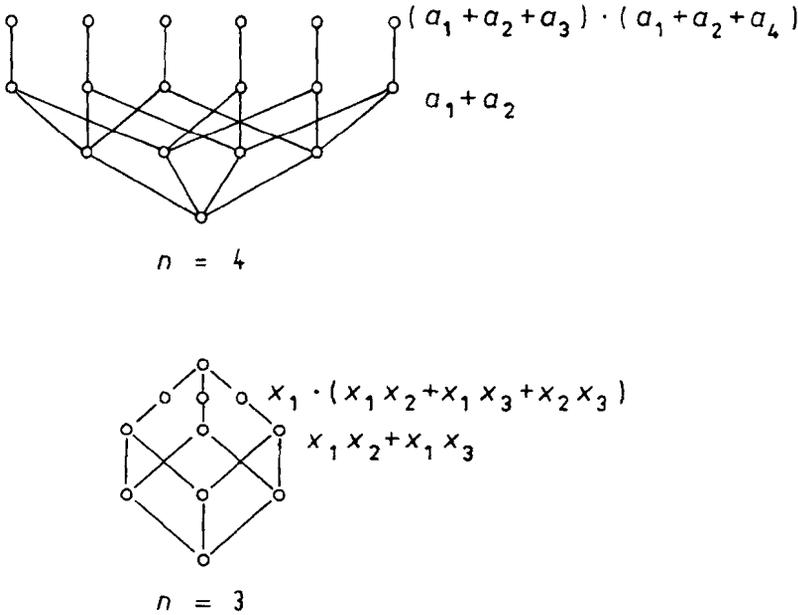


Fig. 1.

LEMMA 2.1. For all i , $w_{i1*} \leq v$.

Proof. Suppose that for some i , $w_{i1*} \not\leq v$. Then since $v^* = w$ we have

$$\prod_i w_i \cdot \prod_k x_k = w \leq w_{i1*} + v.$$

By (W) this implies that for some j , $w_j \leq w_{i1*} + v$, or for some k , $x_k \leq w_{i1*} + v$. Since $w_{i1*} + v \leq w_i$, the first case must hold with $j = i$, and thus $w_{i1*} + v = w_i$. By the refinement property of the canonical joinands described in the last section $\{w_{i1}, w_{i2}, \dots\} \ll \{w_{i1*}, v\}$. However, $w_{i1} \not\leq w_{i1*}$ and $w_{i1} \not\leq v$. This contradiction proves the lemma. □

LEMMA 2.2. Either w has an upper cover or v has a lower cover.

Proof. Suppose not. Then, by Lemma 1.1, no generator is a canonical meetand of w , i.e., $w = \prod_i \sum_j w_{ij}$, and dually $v = \sum_r \prod_s v_{rs}$.

Since, by Theorem 1.3, all the canonical joinands of w_i , except w_{i1} , lie below w , $w_i = w_{i1} + v = w_{i1} + \sum v_r$. It follows that

$$\{w_{i2}, w_{i3}, \dots\} \ll \{v_1, v_2, \dots\} \tag{*}$$

for all i .

For any pair i, r we have

$$\prod_s v_{rs} = v_r \leq w_i = \sum_j w_{ij}.$$

We wish to apply (W) to this inclusion. Suppose $v_r \leq w_{ij}$ for some $j > 1$. By (*) $w_{ij} \leq v_r$

for some t , whence $r = t$ and $v_r = w_{ij} \in J(w)$. Thus, by Theorem 1.2, v_r is completely join irreducible and $v \succ v \cdot \kappa(v_r)$, contrary to our assumption. Hence $v_r \not\leq w_{ij}$ for $j > 1$, and dually $v_{rs} \not\leq w_i$ for $s > 1$. We conclude by (W) that for any pair i, r either $v_{r1} \leq w_i$ or $v_r \leq w_{i1}$.

Assume that $v_r \leq w_{i1}$. Then $v_r \neq w_{i1}$, since $v_r \leq w$, so $v_r \leq w_{i1*}$. On the other hand, $w_{i1*} \leq v$, by Lemma 2.1. Hence, $w \cdot w_{i1} = w_{i1*} \leq v = \sum_t v_t$. Applying (W), we find that $w_{i1*} \leq v_t$ for some t . It follows that $t = r$ and $w_{i1*} = v_r$. By duality, if $v_{r1} \leq w_i$ then $v_{r1*} = w_i$. Thus, we have that for any pair i, r either $v_{r1*} = w_i$ or $w_{i1*} = v_r$.

Now let $i = 1$ and let r vary. Clearly $w_{11*} = v_r$ for at most one r , so there exists t_1 such that $v_{r1*} = w_1$ for $r \neq t_1$. Likewise, there exists t_2 such that $v_{r1*} = w_2$ for $r \neq t_2$. Since $w_1 \neq w_2$, it follows that there are only two v_r 's, i.e., $v = v_1 + v_2$ canonically.

From the case $i = 1, r = 1$ we get either (a) $v_{11*} = w_1$ or (b) $v_1 = w_{11*}$. From $i = 1, r = 2$ we get either (c) $v_{21*} = w_1$ or (d) $v_2 = w_{11*}$, and from $i = 2, r = 1$ either (e) $v_{11*} = w_2$ or (f) $v_1 = w_{21*}$. It follows that either (a) and (f) hold, or else (b) and (c) hold. By duality, we may assume the latter, i.e., $v_1 = w_{11*}$ and $v_{21*} = w_1$.

Note that $w_{11} \not\leq v_{21}$. For otherwise, (b) would imply $v_1 \leq w \cdot w_{11} \leq w \cdot v_{21} = v_2$, a contradiction.

Now consider $w_1 = w + w_{11} = w + v_{21}$ (by (c)). By (SD_{\vee}) this implies $w_1 = w + w_{11} \cdot v_{21} = w$ since $w_{11} \not\leq v_{21}$ and $w_{11*} \leq v \leq w$. This final contradiction proves the lemma. \square

Returning to the proof of the Theorem, we again assume that $w \succ v$ is a singular cover. By Lemma 2.2 and duality we may assume that there is an element u covering w . Since v is meet irreducible, the interval u/v is a three element chain. By Theorem 1.5 there is a completely meet irreducible element q such that $w = u \cdot q$ canonically. Moreover, if u_1 denotes the unique canonical joinand of u not below w , then $u_1 = \kappa'(q)$. This situation is diagrammed in Figure 2. The solid lines are used to indicate coverings.

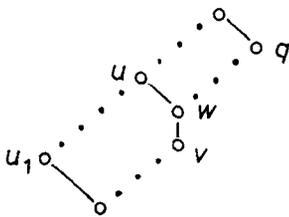


Fig. 2.

Now $u = u_1 + v = u_1 + \sum v_r$. Since $u = \sum u_i$ canonically we obtain from the refinement property of Section 2:

$$\{u_i : i > 1\} \ll \{v_1, v_2, \dots\}. \tag{**}$$

Note that there is an r such that $v_r \not\leq u_1$. For otherwise we would have $v \leq u_1 \leq u$, which, since $u_1 \not\leq w$, implies $u_1 = u$. Since u_1 is a canonical joinand of u , $u = u_1$ implies

that u is either a generator of a proper meet. This however contradicts Theorem 1.5 which says that u is join reducible.

Fix an r such that $v_r \not\leq u_1$. Then $\Pi_s v_{rs} = v_r \leq u = \Sigma u_i$. Applying (W) to this inclusion we get four cases.

Case 1. $v_r \leq u_1$. This cannot occur by the choice of r .

Case 2. $v_r \leq u_i$ for some $i > 1$. By (**) we also have $u_i \leq v_s$ for some s , whence $r = s$ and $v_r = u_i \in J(w)$. Since w is completely join irreducible, v_r must be completely join irreducible by Theorem 1.2, and so $v > v \cdot \kappa(v_r)$ by Lemma 1.1. Hence $u > w > v > v \cdot \kappa(v_r)$ is a covering chain of length 3. By Theorem 9.2 of [9], such chains occur only in the connected component of $FL(X)$ containing 0 or 1, where the known singular covers in $FL(X)$ also occur. Thus we may assume that $v_r \not\leq u_i$ for all $i > 1$.

Case 3. $v_{rs} \leq u$ for some $s > 1$. Now $v_{rs} > v$ (as $s > 1$) and v_{rs} is meet irreducible, so this implies $v_{rs} = u$. But $v_{rs} \in M(v)$ is completely meet irreducible, i.e., $u = v_{rs}$ has an upper cover u^* . Again this gives a covering chain of length three, so we may assume that this case does not occur.

Case 4. $v_{r1} \leq u$. By the dual of Lemma 2.1 we have $v_{r1}^* \geq w$, so $w \leq v_{r1}^* \leq u$. This implies that $v_{r1}^* = u$, since $v_{r1}^* = w$ would imply that $w = v_{r1} + v$ is a proper join, a contradiction.

Let u_2 be the canonical joinand of u not below v_{r1} , so that $v_{r1} = u \cdot \kappa(u_2)$. Now v_{r1} , being a canonical meetand of v_r , is meet irreducible, so in fact $v_{r1} = \kappa(u_2)$. We claim that $u_1 \neq u_2$. Clearly $v_{r1} \neq w$ since w is meet reducible and v_{r1} is meet irreducible. Also $v_{r1} \not\leq v$ by the definition of canonical form. Since every element properly below w is below v , the above implies that $v_{r1} \not\leq w$. Now if $v_{r1} \leq q$ then $v_{r1} \leq q \cdot u = w$, a contradiction. Thus $\kappa(u_2) = v_{r1} \not\leq q$, but $\kappa(u_1) = q$, showing that $u_1 \neq u_2$.

Now apply the dual of Theorem 1.4. Since $u = v_{r1}^*$, the canonical joinands of u are precisely u_2 and the canonical joinands of v_{r1} not below $\kappa'(v_{r1}) = u_2$. Thus each $u_i, i \neq 2$, and especially u_1 , is a canonical joinand of v_{r1} .

Hence,

$$v_{r1} > v_{r1} \cdot \kappa(u_1) = v_{r1} \cdot q = v_{r1} \cdot u \cdot q = v_{r1} \cdot w = v_r.$$

Since $v_r \leq v$, we have shown that the connected component of u in $FL(X)$ contains at least the configuration diagrammed in Figure 3, where solid lines denote coverings.

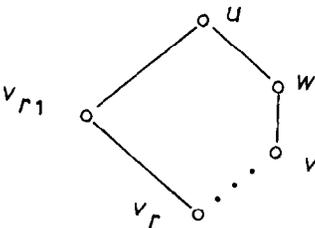


Fig. 3.

By checking the main theorem of [6], which characterizes connected components in

free lattices, we see that the above configuration is never part of a connected component.

This final contradiction completes the proof of the theorem.

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