Representing Finite Lattices as Congruence Lattices

William DeMeo, Ralph Freese, Peter Jipsen

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The Problem

Theorem (Grätzer-Schmidt)

Every algebraic (so every finite) lattice is isomorphic to $\text{Con} (A)$ for some (unary) algebra $A$. 
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Since $\text{Con}(A) = \text{Con}(\langle A, \text{Pol}_1(A) \rangle)$, we assume all algebras are unary.
Possible representation properties for a finite lattice \( L \):

(P1) \( L \) is isomorphic to the congruence lattice of some finite algebra \( \langle A, F \rangle \).

(P2) \( L \) is isomorphic to the congruence lattice of some finite algebra \( \langle A, F \rangle \) where the all nonconstant operations are permutations.

(P3) \( L \) is isomorphic to the congruence lattice of some finite algebra \( \langle A, F \rangle \) where the nonconstant operations generate a transitive permutation group.

(P4) \( L \) is isomorphic to an interval in the lattice of subgroups of a finite group.

\( \text{P4} \Leftrightarrow \text{P3} \Rightarrow \text{P2} \Rightarrow \text{P1} \)
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$(P4) \Leftrightarrow (P3) \Rightarrow (P2) \Rightarrow (P1)$
Let $H$ be a subgroup of $G$. Let

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Then $\text{Con } A \cong [H, G]$, the interval in the subgroup lattice.
Theorem (1980)

(P1) holds for all lattices iff (P4) holds for all lattice.
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The size of a representation $L \cong \text{Con}(A)$ is $|A|$. For $H \leq G$ the size in (P4) is $[G : H]$, the number of left $H$-cosets of $G$. 

Example. The minimum size for $L_6$ is 6:

$$f(x) = \begin{cases} 
2 & \text{if } x = 1 \\
2 & \text{if } x = 2 \\
1 & \text{if } x = 3 \\
5 & \text{if } x = 4 \\
5 & \text{if } x = 5 \\
4 & \text{if } x = 6 
\end{cases}$$

$$g(x) = \begin{cases} 
3 & \text{if } x = 1 \\
4 & \text{if } x = 2 \\
4 & \text{if } x = 3 \\
0 & \text{if } x = 4 \\
1 & \text{if } x = 5 \\
1 & \text{if } x = 6 
\end{cases}$$

$$h(x) = \begin{cases} 
4 & \text{if } x = 1 \\
5 & \text{if } x = 2 \\
3 & \text{if } x = 3 \\
4 & \text{if } x = 4 \\
5 & \text{if } x = 5 \\
3 & \text{if } x = 6 
\end{cases}$$

Pálfy and Aschbacher have found groups $H \leq G$ representing this lattice. But Pálfy’s example has $G = A_{11}$ and $|H| = 55$, so the size is $9! = 362,880$. 

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**Example.** The minimum size for $L_6$ is 6:

<table>
<thead>
<tr>
<th>$B_6$</th>
<th>0 1 2 3 4 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>2 2 1 5 5 4</td>
</tr>
<tr>
<td>$g(x)$</td>
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**Example.** The minimum size for \( L_6 \) is 6:

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\hline
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B_6 & f(x) & 2 & 2 & 1 & 5 & 5 & 4 \\
g(x) & 3 & 4 & 4 & 0 & 1 & 1 \\
h(x) & 4 & 5 & 3 & 4 & 5 & 3 \\
\hline
\end{array}
\]

Pálfy and Aschbacher have found groups \( H \leq G \) representing this lattice. But Pálfy’s example has \( G = A_{11} \) and \( |H| = 55 \), so the size is \( 9! = 362880 \).
**Moral:** Finding a representation with groups, (P4), may be much harder (and much bigger) than finding a (P1) representation.
New representable lattices from old

- all distributive lattices
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interval sublattices
direct products (Jiří Tůma, 1986)
ordinal sums (Ralph McKenzie, 1984; John Snow, 2000)
parallel sums (John Snow, 2000)
sublattices of representable lattices obtained as a union of a filter and an ideal (John Snow, 2000)
overalgebras (DeMeo, 2013)
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- overalgebras (DeMeo, 2013)
(A) \( L \) is simple.

(B) For each \( x \neq 0 \) in \( L \), there are elements \( y \) and \( z \) such that

\[
x \lor y = x \lor z = 1 \quad \text{and} \quad y \land z = 0.
\]

(C) \( |L| \neq 2 \) and each element of \( L \) that is not an atom or 0

contains at least four atoms.

Theorem

If \( L \) satisfies \( (A) \) and \( (B) \) then \( L \) satisfies \( (P1) \Rightarrow (P2) \).

If \( L \) satisfies \( (A) \), \( (B) \) and \( (C) \) then \( L \) satisfies \( (P1) \Rightarrow (P3) \).
Pálfy-Pudlák Conditions

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- *If $L$ satisfies (A) and (B) then $L$ satisfies (P1) $\Rightarrow$ (P2).*
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- If \( L \) satisfies (A), (B) and (C) then \( L \) satisfies (P1) \( \Rightarrow \) (P3).
McKenzie’s variants

(B′) If $\varphi : L \to L$ is any meet-preserving map such that $\varphi(x) > x$ for $x \neq 1$, then $\varphi(x) = 1$ for all $x$. 

(B′′) The coatoms of $L$ meet to 0.

(B) $\Rightarrow$ (B′′) $\Rightarrow$ (B′).

Theorem

If $L$ satisfies (A) and (B′) (or (B′′)) then a minimal representation of $L$ witnesses that $L$ satisfies (P2). So, if (A) and (B′) hold and $L \sim = \langle A, F \rangle$ is minimal, then $F$ consists of permutations and constants.
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\( (B') \) If \( \varphi : L \to L \) is any meet-preserving map such that \( \varphi(x) > x \) for \( x \neq 1 \), then \( \varphi(x) = 1 \) for all \( x \).

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If \( L \) satisfies (A) and (B′) (or (B″)) then a minimal representation of \( L \) witnesses that \( L \) satisfies (P2). So, if (A) and (B′) hold and \( L \cong \langle A, F \rangle \) is minimal, then \( F \) consists of permutations and constants.
Representations by intransitive groups

Suppose $A = \langle A, G \rangle$ is a $G$-set and let $A_i = \langle A_i, G \rangle$, $i < k$, be the minimal subalgebras of $A$; i.e. each set $A_i$ is an orbit, or one-generated subuniverse, of $A$. Define congruences on $A$ by the partitions

$$\tau = |A_0|A_1| \cdots |A_{k-1}| \quad \text{(the blocks are the orbits)}$$
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We call $\tau$ the **intransitivity congruence**;

**Theorem**

Let $\theta \in \text{Con } (A)$, where $A = \langle A, G \rangle$ and $G$ is a group. Then
\[ \tau = |A_0|A_1| \cdots |A_{k-1}| \] (the blocks are the orbits)

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5. If \( \theta \wedge \tau < \tau \) then \( \theta \leq \gamma_i \) for some \( i \).
6. If \( k > 1 \) and \( |A_i| = 1 \) for all \( i \) except 0 then every coatom of \( \text{Con} (A) \) lies above \( \tau \).
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$L_{14}$ satisfies (A) and (B'') so a minimal representation is permutational.

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**Examples: \( L_{14} \)**

\[ \begin{array}{c}
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- Is \( L_{14} \) representable? (Yes: as \([H, A_6]\) with \([A_6 : H] = 90\))
- Is this a minimum representation? (Don’t know)
Examples: $L_{15}$, (the dual of $L_{14}$)
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$\tau$
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Example

$L_{15} \cong \text{Con} \langle \{0, 1, 2, 3\}, G \rangle$, $G$ the group generated by the double transposition $0 \leftrightarrow 1, 2 \leftrightarrow 3$. 
Examples: $L_{15}$, (the dual of $L_{14}$)

Example

- $L_{15} \cong \text{Con} \langle \{0, 1, 2, 3\}, G \rangle$, $G$ the group generated by the double transposition $0 \leftrightarrow 1, 2 \leftrightarrow 3$.
- $L_{14} \cong L_{15}^d$, which again proves $L_{14}$ is representable.
Examples: $L_4$. 

![Diagram of $L_4$ lattice]
Examples: $L_4$.

Example

$L_4$ satisfies $(B'')$ but not (A) so minimal representations need not be permutational.
Examples: \( L_4 \).

\[ \tau \]

Example

\( L_4 \) satisfies (B'') but not (A) so minimal representations need not be permutational. In fact

\( L_4 \cong \langle \{0, 1, 2, 3\}, f, g \rangle \), where

\[
\begin{array}{c|cccc}
B_4 & 0 & 1 & 2 & 3 \\
\hline
f(x) & 1 & 0 & 3 & 2 \\
g(x) & 0 & 0 & 2 & 2 \\
\end{array}
\]
Example

- $L_4$ satisfies (B’’) but not (A) so minimal representations need not be permutational.
- But $L_4$ does have an intransitive representation on 6:

$$
\begin{array}{c|ccccc}
B'_4 & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
f(x) & 1 & 2 & 0 & 4 & 5 & 3 \\
g(x) & 0 & 2 & 1 & 3 & 5 & 4 \\
\end{array}
$$
Examples: \( L_{19} \), a harder example:
Examples: $L_{19}$, a harder example:

Lemma

Let $A = \langle A, G \rangle$ be a finite algebra, where $G$ is an intransitive group of permutations on $A$. Suppose the intransitivity congruence $\tau$ is a coatom. Then there do not exist congruences $0_A < \psi < \theta$ in $\text{Con} (A)$ with $\theta \land \tau = 0_A$. 
Proof

Lemma

Let $\mathbf{A} = \langle A, G \rangle$ be a finite algebra, where $G$ is an intransitive group of permutations on $A$. Suppose the intransitivity congruence $\tau$ is a coatom. Then there do not exist congruences $0_A < \psi < \theta$ in $\text{Con}(A)$ with $\theta \land \tau = 0_A$.

Proof.

Since $\tau$ is a coatom, there are exactly two orbits; call them $B$ and $C$. Since $\theta \land \tau = 0_A$, if $(x, y) \in \theta$ then $x = y$ or one is in $B$ and the other is in $C$. So $\theta$ defines a bipartite graph between $B$ and $C$. Since $G$ acts transitively on both $B$ and $C$, this graph corresponds to a bijection between $B$ and $C$. The same applies to $\psi$. But equivalence relations corresponding to such graphs cannot be comparable.
Theorem

All lattices with at most 7 elements can be represented, with the one possible exception of $L_{10}$:
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If $L_{10} \cong \langle A, F \rangle$, then $F$ generates a transitive group on $A$. 
**Theorem**

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If $L_{10} \cong \langle A, F \rangle$, then $F$ generates a transitive group on $A$.

**Proof.**

$L_{10}$ satisfies (A) and (B’’). By part (5) of the intransititivity theorem, it cannot be represented with an intransitive group.
Closure Method
Finding Reps: Methods and Algorithms

- Closure Method
- Overallgebras
Finding Reps: Methods and Algorithms

- Closure Method
- Overalgebras
- Ideal-Filter
Finding Reps: Methods and Algorithms

- Closure Method
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- Duality
Finding Reps: Methods and Algorithms

- Closure Method
- Overalgebras
- Ideal-Filter
- Duality
- Group Methods (GAP)
Closure Method to find a Representation of $L$

(1) Search through $\text{Eq}(X_k)$, $k = 2, 3, ...$ finding sublattices isomorphic to $L$.

(2) For each sublattice $L \sim L' \leq \text{Eq}(X_k)$ found, find the unary polymorphs of the members of $L'$; that is, calculate the set $F$ of all unary operations on $X_k$ which respect all $\theta \in L'$.

(3) For $F$ found in the previous step, test if $\text{Con}(\langle X_k, F \rangle) = L'$. If so then $A = \langle X_k, F \rangle$ is a minimal representation. Otherwise continue the search.
Closure Method to find a Representation of \( L \)

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(3) For $F$ found in the previous step, test if $\text{Con}(\langle X_k, F \rangle) = L'$. If so then $A = \langle X_k, F \rangle$ is a minimal representation. Otherwise continue the search.
(a) **Find a small presentation of L:**
The procedure can be sped up by first finding a presentation of L with the minimal number of generators. Besides speeding up the search in Eq(k), it is enough in calculating the unary polymorphs to respect the generators.
(b) **Subdirect Decompositions:**

Subdirect decompositions can be used to speed up finding unary polymorphs. For example, if \( \theta_0, \theta_1 \in L' \leq \text{Eq}(X_k) \) with \( \theta_0 \land \theta_1 = 0 \), then \( X_k \) is naturally embedded into \( X_k/\theta_0 \times X_k/\theta_1 \). Since the operations in a direct product are component-wise, this cuts the search space of possible unary polymorphs from \( k^k \) down to \( r^r s^s \), where \( r \) and \( s \) are the number of blocks in \( \theta_0 \) and \( \theta_1 \).
(c) **Uniform Equivalence Relations:**
If it can be shown that the algebra of a minimal representation of $L$ has a transitive permutation group for its nonconstant unary polynomials, then we can restrict our search in $\text{Eq}(k)$ to uniform equivalence relations. Moreover the search for unary polymorphs can be restricted to permutations.
(d) **Small generating set for the operations:**
Of course if $F' \subseteq F$ is a set of generators for the moniod $F$, we can take $A = \langle X_k, F' \rangle$. 
Nondist., linearly indec., small lattices

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Method: overalgebras
\[ L_4 \quad \begin{array}{c|cccc} B_4 & 0 & 1 & 2 & 3 \\ \hline f(x) & 1 & 0 & 3 & 2 \\ g(x) & 0 & 0 & 2 & 2 \end{array} \]

\[ L_5 \quad \begin{array}{c|cccccccccccc} B_5 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline f(x) & 1 & 2 & 3 & 4 & 5 & 0 & 7 & 8 & 9 & 10 & 11 & 6 \\ g(x) & 6 & 1 & 1 & 0 & 9 & 8 & 7 & 0 & 5 & 4 & 3 & 2 & 1 \\ h(x) & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \end{array} \]

\[ L_6 \quad \begin{array}{c|ccccc} B_6 & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline f(x) & 2 & 2 & 1 & 5 & 5 & 4 \\ g(x) & 3 & 4 & 4 & 0 & 1 & 1 \\ h(x) & 4 & 5 & 3 & 4 & 5 & 3 \end{array} \]
Method: overalgebras
No finite algebra known with this as its congruence lattice.

A finite algebra with 108 elements known.

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Method: overalgebras

Upper interval in Sub($A_6$),
algebra of size 90

Upper interval in Sub($C_2 \cdot A_6$)
algebra of size 180
Method: filter-ideal in Sub($A_4$)

Dual of 19, no explicit small representation known

Method: filter-ideal in SmallGroup(216,153) in GAP
$$\begin{array}{c|cccccccc} \mathbf{B}_{21} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \\
 f(x) & 3 & 3 & 4 & 8 & 8 & 2 & 2 & 3 & 4 \\ g(x) & 0 & 0 & 6 & 1 & 1 & 0 & 0 & 5 & 6 \\ h(x) & 4 & 5 & 5 & 7 & 8 & 8 & 7 & 4 & 4 \\ \end{array}$$

Dual of 23, no explicit small representation known

$$\begin{array}{c|cccc} \mathbf{B}_{23} & 0 & 1 & 2 & 3 \\ \hline \\
 f(x) & 0 & 1 & 0 & 1 \\ g(x) & 1 & 1 & 3 & 3 \\ h(x) & 3 & 2 & 3 & 2 \\ k(x) & 4 & 1 & 5 & 3 \\ \end{array}$$

$$\begin{array}{c|cccc} \mathbf{B}_{24} & 0 & 1 & 2 & 3 \\ \hline \\
 f(x) & 1 & 1 & 2 & 2 \\ g(x) & 2 & 3 & 3 & 2 \\ \end{array}$$
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\[
\begin{array}{c|cccc}
B_{25} & 0 & 1 & 2 & 3 & 4 \\
\hline
f(x) & 0 & 0 & 2 & 2 & 2 \\
g(x) & 0 & 1 & 0 & 1 & 1 \\
h(x) & 1 & 1 & 4 & 4 & 4 \\
k(x) & 2 & 3 & 2 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
B_{26} & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
f(x) & 1 & 0 & 3 & 2 & 0 & 2 \\
g(x) & 4 & 4 & 5 & 5 & 1 & 3 \\
h(x) & 0 & 0 & 0 & 0 & 1 & 1 \\
k(x) & 3 & 5 & 3 & 5 & 3 & 3 \\
\end{array}
\]
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**Method:** overalgebras

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<td>h(x)</td>
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<tr>
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<tr>
<td>m(x)</td>
<td>11 12 2 13 14 15 12 2 13 14 15 11 12 13 14 15</td>
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**Method:** overalgebras

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\begin{align*}
\text{L}_{29} & \quad B_{29} \quad 0 \ 1 \ 2 \ 3 \ 4 \\
& \quad f(x) \quad 1 \ 0 \ 3 \ 2 \ 2 \\
& \quad g(x) \quad 2 \ 4 \ 2 \ 4 \ 3 \\
\text{L}_{30} & \quad B_{30} \quad 0 \ 1 \ 2 \ 3 \ 4 \\
& \quad f(x) \quad 0 \ 3 \ 4 \ 3 \ 4 \\
& \quad g(x) \quad 2 \ 2 \ 1 \ 4 \ 3 \\
\text{L}_{31} & \quad B_{31} \quad 0 \ 1 \ 2 \ 3 \ 4 \\
& \quad f(x) \quad 0 \ 1 \ 1 \ 0 \ 0 \\
& \quad g(x) \quad 1 \ 1 \ 2 \ 2 \ 2 \\
& \quad h(x) \quad 3 \ 2 \ 2 \ 4 \ 4 \\
\text{L}_{32} & \quad B_{32} \quad 0 \ 1 \ 2 \ 3 \ 4 \\
& \quad f(x) \quad 0 \ 1 \ 1 \ 3 \ 3 \\
& \quad g(x) \quad 1 \ 2 \ 2 \ 4 \ 4 \\
& \quad h(x) \quad 3 \ 3 \ 4 \ 3 \ 4
\end{align*}
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\[
\begin{array}{c|cccccccccccccccc}
\mathbf{B}_{33} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\hline
f(x) & 1 & 3 & 2 & 0 & 9 & 11 & 10 & 8 & 13 & 15 & 14 & 12 & 5 & 7 & 6 & 4 \\
g(x) & 11 & 8 & 10 & 9 & 7 & 4 & 6 & 5 & 15 & 12 & 14 & 13 & 3 & 0 & 2 & 1 \\
h(x) & 14 & 15 & 12 & 13 & 10 & 11 & 8 & 9 & 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\
\end{array}
\]