Modular Lattices Embedded into Congruence Lattices of Algebras in almost all Varieties

Ralph Freese

http://math.hawaii.edu/∼ralph/

http://uacalc.org/

https://github.com/UACalc/

AMS Special Session: Algebras and Algorithms, JMM 2020

Goal

$$\mathbf{Con}\ \mathcal{V} = \{\mathbf{Con\ A} : \mathbf{A} \in \mathcal{V}\}$$

$$\boldsymbol{S}\,\mathbf{Con}\ \mathcal{V} = \text{sublattices of these lattice}$$

Goal

$$\begin{aligned}
\textbf{Con}\ \mathcal{V} &= \{\textbf{Con}\ \mathbf{A} : \mathbf{A} \in \mathcal{V}\}\\
\boldsymbol{S}\,\textbf{Con}\ \mathcal{V} &= \text{sublattices of these lattice}\\
\boldsymbol{HS}\ \textbf{Con}\,\mathcal{V} &= \boldsymbol{HSP}\,\textbf{Con}\ \mathcal{V} = \boldsymbol{V}\,\textbf{Con}\ \mathcal{V}\\
&= \text{the congruence variety of } \mathcal{V}
\end{aligned}$$

Goal

$$\begin{aligned}
\mathbf{Con}\ \mathcal{V} &= \{\mathbf{Con}\ \mathbf{A} : \mathbf{A} \in \mathcal{V}\} \\
\boldsymbol{S}\,\mathbf{Con}\ \mathcal{V} &= \text{sublattices of these lattice} \\
\boldsymbol{HS}\ \mathbf{Con}\,\mathcal{V} &= \boldsymbol{HSP}\,\mathbf{Con}\ \mathcal{V} = \boldsymbol{V}\ \mathbf{Con}\ \mathcal{V} \\
&= \text{the congruence variety of } \mathcal{V}
\end{aligned}$$

Goal: Show every modular lattice you have ever drawn (and several you haven't) is in $\boldsymbol{S}\,\mathbf{Con}\,\mathcal{V}$, for most $\mathcal{V}$.

A Useful Construction

Congruences as Algebras and their Congruence Lattices

A is an algebra and α a congruence on **A**.

A Useful Construction

Congruences as Algebras and their Congruence Lattices

A is an algebra and α a congruence on **A**.

α is a subalgebra of $\mathbf{A}^2$ denoted $\mathbf{A}(\alpha)$ or $\mathbf{A} \times_\alpha \mathbf{A}$.

A Useful Construction

Congruences as Algebras and their Congruence Lattices

A is an algebra and α a congruence on **A**.

α is a subalgebra of $\mathbf{A}^2$ denoted $\mathbf{A}(\alpha)$ or $\mathbf{A} \times_\alpha \mathbf{A}$. For $\theta \in \mathbf{Con}\ \mathbf{A}$, define $\theta_i \in \mathbf{Con}(\mathbf{A}(\alpha))$, $i = 0, 1$, by

A Useful Construction

Congruences as Algebras and their Congruence Lattices

A is an algebra and α a congruence on **A**.

α is a subalgebra of $\mathbf{A}^2$ denoted $\mathbf{A}(\alpha)$ or $\mathbf{A} \times_\alpha \mathbf{A}$. For $\theta \in \mathbf{Con\,A}$, define $\theta_i \in \mathbf{Con}(\mathbf{A}(\alpha))$, $i = 0, 1$, by

$$\theta_i = \{\langle\langle a_0, a_1\rangle, \langle b_0, b_1\rangle\rangle \in A(\alpha) \times A(\alpha) : \langle a_i, b_i\rangle \in \theta\}.$$

η_0 and η_1 are the kernels of the projections (not 0_0 and 0_1).

A Useful Construction . . .

Lemma

Let α, θ and ψ be congruences on **A**. *With notation as above:*

(i) *The map $\langle a_0, a_1 \rangle \mapsto \langle a_1, a_0 \rangle$ defines an automorphism of $\mathbf{A}(\alpha)$ which interchanges θ_0 and θ_1.*

(ii) *The map $\theta \mapsto \theta_i$ is a lattice isomorphism of* **Con A** *onto the interval $\mathbf{I}[\eta_i, 1_{\mathbf{A}(\alpha)}]$ of* **Con** $\mathbf{A}(\alpha)$, *for $i = 0, 1$. So $(\theta \vee \psi)_i = \theta_i \vee \psi_i$ and dually.*

(iii) *If $\psi \geq \alpha$ then $\psi_0 = \psi_1$.*

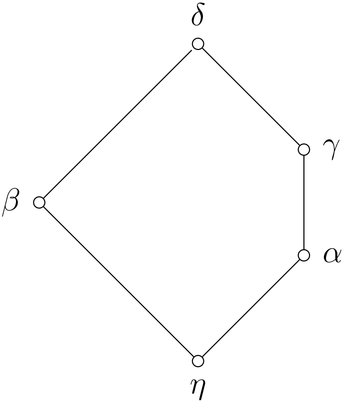
(iv) $\eta_0 \vee \eta_1 = \alpha_0$ $(= \alpha_1)$.

(v) *η_0 and η_1 permute.*

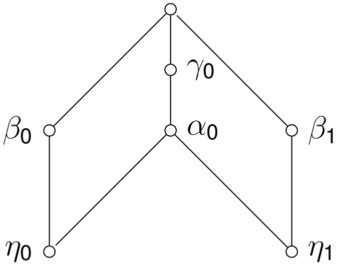
(vi) $(\theta_0 \wedge \theta_1) \vee \eta_0 = \theta_0$; *in fact* $\theta_0 = \eta_0 \circ (\theta_0 \wedge \theta_1) \circ \eta_0$.

Proof. Easy calculations.

A Useful Construction

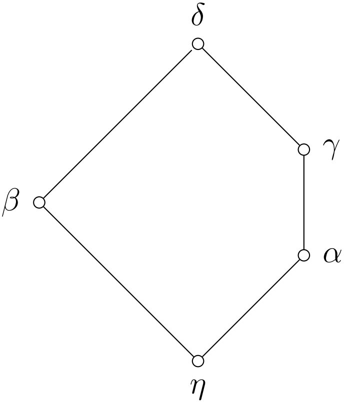


Con A

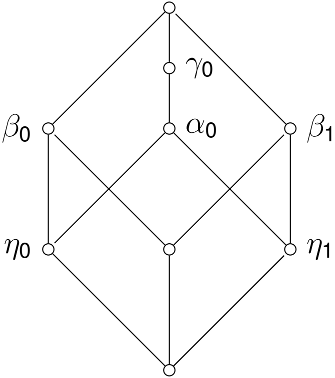


Con A(α)

A Useful Construction

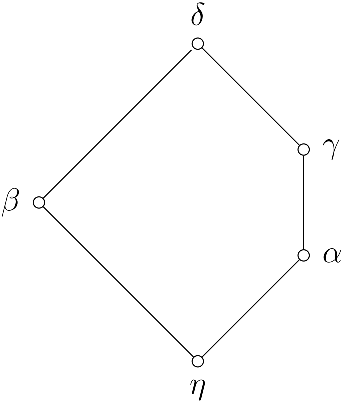


Con A

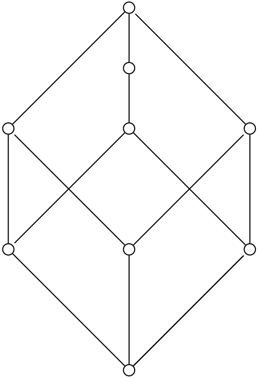


Con A$(\alpha) =$ **L**$_{14}$

A Useful Construction



Con A



Con A$(\alpha) =$ **L**$_{14}$

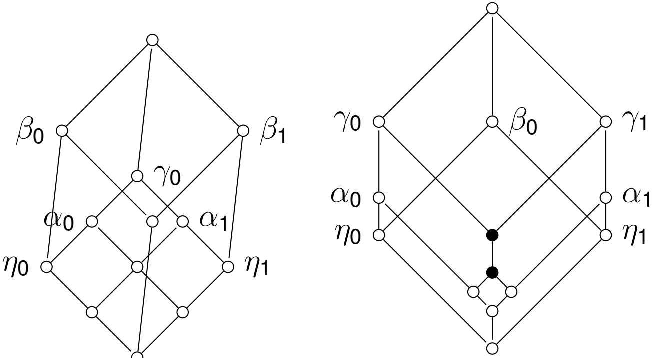
So if $\mathcal{V}$ is not CM there is **A** $\in \mathcal{V}$ with **L**$_{14}$ as a sublattice.

What about **Con A**(β) and **Con A**(γ)?

These are not uniquely determined but

What about **Con A**(β) and **Con A**(γ)?

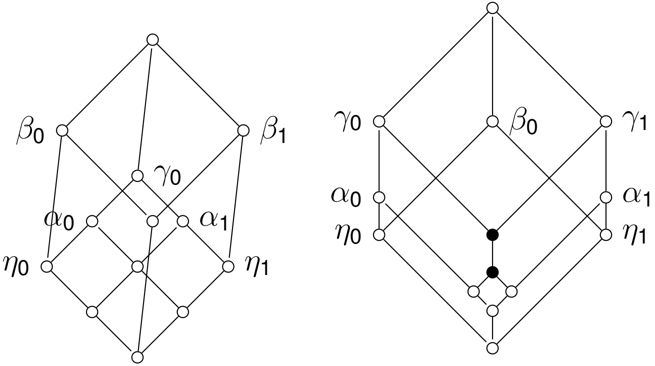
These are not uniquely determined but if $\mathbf{A} = \mathbf{P}$ is Polin's algebra:



Con P(γ) **Con P**(β)

What about **Con A**(β) and **Con A**(γ)?

These are not uniquely determined but if $\mathbf{A} = \mathbf{P}$ is Polin's algebra:



Con P(γ) **Con P**(β)

Con P(β) is SI and projective so is in $\boldsymbol{S}$ **Con** $\mathcal{V}$ if it is not CM.

Uses of the “Useful Construction"

- If $\mathbf{A} \in \mathcal{V}$, $\mathcal{V}$ CM, then **Con** $A \neq \mathbf{M}_7$.

Uses of the “Useful Construction"

- If $\mathbf{A} \in \mathcal{V}$, $\mathcal{V}$ CM, then $\mathbf{Con}\ A \neq \mathbf{M}_7$.
- (Nation) $\mathcal{V}$ not CM implies $\mathbf{L}_{14} \in \boldsymbol{S}\,\mathbf{Con}\ \mathcal{V}$.

Uses of the "Useful Construction"

- If $\mathbf{A} \in \mathcal{V}$, $\mathcal{V}$ CM, then $\mathbf{Con}\, A \neq \mathbf{M}_7$.
- (Nation) $\mathcal{V}$ not CM implies $\mathbf{L}_{14} \in \boldsymbol{S}\,\mathbf{Con}\,\mathcal{V}$.
- (with Jónsson) If $\mathcal{V}$ is CM, it is congruence Arguesian.

Uses of the “Useful Construction"

- If $\mathbf{A} \in \mathcal{V}$, $\mathcal{V}$ CM, then $\mathbf{Con}\ A \neq \mathbf{M}_7$.
- (Nation) $\mathcal{V}$ not CM implies $\mathbf{L}_{14} \in \boldsymbol{S}\,\mathbf{Con}\ \mathcal{V}$.
- (with Jónsson) If $\mathcal{V}$ is CM, it is congruence Arguesian.
- (with A. Day) $\mathcal{V}$ not CM implies $\boldsymbol{HS}\,\mathbf{Con}\ \mathcal{P} \subseteq \boldsymbol{HS}\,\mathbf{Con}\ \mathcal{V}$.

Uses of the “Useful Construction"

- If $\mathbf{A} \in \mathcal{V}$, $\mathcal{V}$ CM, then $\mathbf{Con}\ A \neq \mathbf{M}_7$.
- (Nation) $\mathcal{V}$ not CM implies $\mathbf{L}_{14} \in \boldsymbol{S}\,\mathbf{Con}\ \mathcal{V}$.
- (with Jónsson) If $\mathcal{V}$ is CM, it is congruence Arguesian.
- (with A. Day) $\mathcal{V}$ not CM implies $\boldsymbol{HS}\,\mathbf{Con}\ \mathcal{P} \subseteq \boldsymbol{HS}\,\mathbf{Con}\ \mathcal{V}$.
- $\mathcal{V}$ not CSD implies $\boldsymbol{HS}\,\mathbf{Con}\ \mathcal{M}_p \subseteq \boldsymbol{HS}\,\mathbf{Con}\ \mathcal{V}$, p a prime. ($\mathcal{M}_p =$ vector spaces over $\mathbb{F}_p$.)

Modular Lattices

$\mathcal{V}$ CM but not CD. Then $\exists$ $\mathbf{A} \in \mathcal{V}$ with $\mathbf{M}_3 \leq \mathbf{Con\ A}$

Modular Lattices

$\mathcal{V}$ CM but not CD. Then $\exists$ **A** $\in \mathcal{V}$ with $\mathbf{M}_3 \leq$ **Con A** and $\mathbf{M}_3$ is a cover-preserving sublattice.

Modular Lattices

$\mathcal{V}$ CM but not CD. Then $\exists$ $\mathbf{A} \in \mathcal{V}$ with $\mathbf{M}_3 \leq \mathbf{Con\ A}$ and

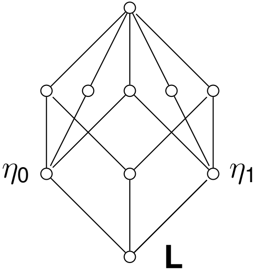
$\mathbf{M}_3$ is a cover-preserving sublattice. This is due Jónsson; the idea of the proof is on the cover of every recent AU.

Modular Lattices

$\mathcal{V}$ CM but not CD. Then $\exists$ $\mathbf{A} \in \mathcal{V}$ with $\mathbf{M}_3 \leq \mathbf{Con\ A}$ and

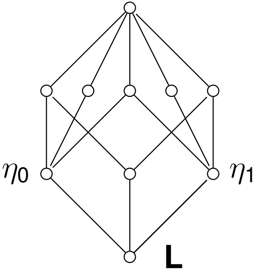
$\mathbf{M}_3$ is a cover-preserving sublattice. This is due Jónsson; the idea of the proof is on the cover of every recent AU. Applying the Useful Construction we get the following configuration which is only partial.

Modular Lattices



The picture of **L** is not complete but we can conclude:

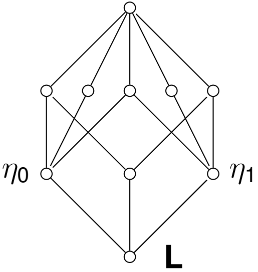
Modular Lattices



The picture of **L** is not complete but we can conclude:

- It has length 3 and is simple (by modularity).
- It is complemented (by Crawley-Dilworth **4.3**).

Modular Lattices



The picture of **L** is not complete but we can conclude:

- It has length 3 and is simple (by modularity).
- It is complemented (by Crawley-Dilworth **4.3**).

Theorem

L *is isomorphic to lattice of subspace of a vectors space of dimension 3, or a projective plane.*

Modular Lattices

We can easily extend this to higher dimensions and conclude:

Theorem

If $\mathcal{V}$ is CM but not CD, there is a p, a prime or 0, so that $\mathbf{S}\,\mathbf{Con}\,\mathcal{V}$ contains $\mathcal{L}_p$, all subspace lattices of all finite dimensional vector spaces over the prime field of characteristic p.

Modular Lattices

We can easily extend this to higher dimensions and conclude:

Theorem

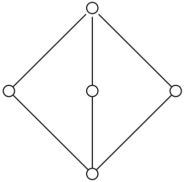
If $\mathcal{V}$ is CM but not CD, there is a p, a prime or 0, so that $\mathbf{S}\,\mathbf{Con}\,\mathcal{V}$ contains $\mathcal{L}_p$, all subspace lattices of all finite dimensional vector spaces over the prime field of characteristic p.

Corollary

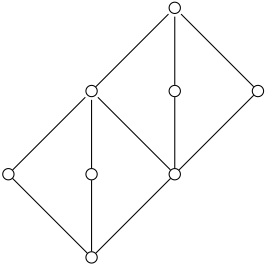
If $\mathcal{V}$ is CM but not CD and $\mathcal{K} = \bigcap_p \mathcal{L}_p$, then

$$\boxed{\mathcal{K} \subseteq \mathbf{S}\,\mathbf{Con}\,\mathcal{V}.}$$

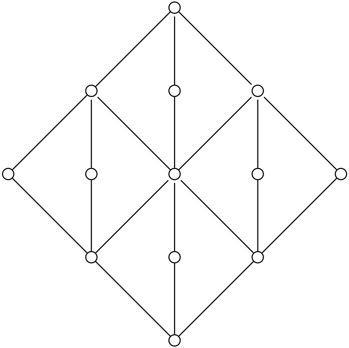
Modular Lattices in $\mathcal{K}$



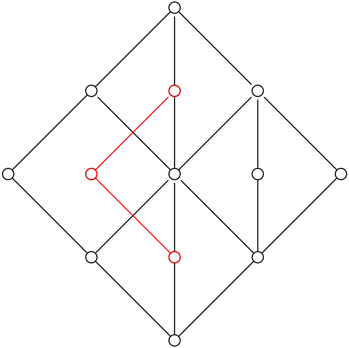
Modular Lattices in $\mathcal{K}$



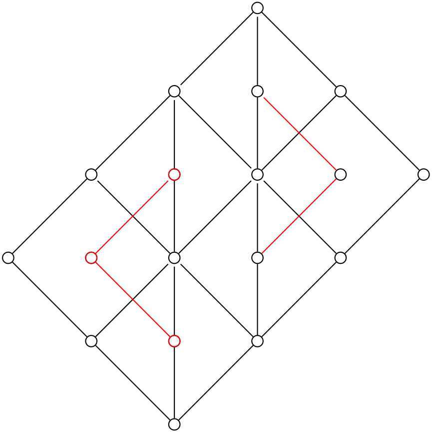
Modular Lattices in $\mathcal{K}$



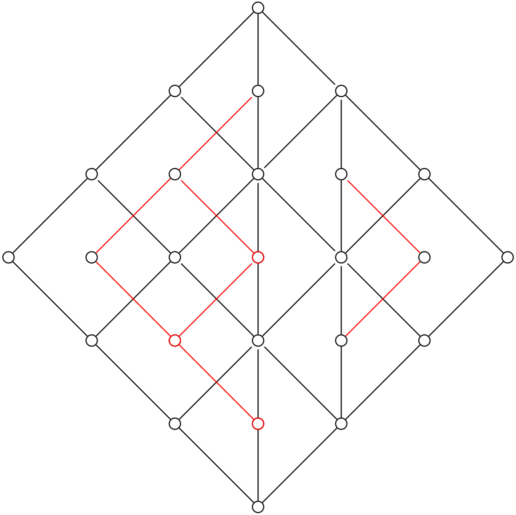
Modular Lattices in $\mathcal{K}$



Modular Lattices in $\mathcal{K}$



Modular Lattices in $\mathcal{K}$



Modular Lattices in $\mathcal{K}$ . . .

All modular lattices drawn above are **isometrically** embedded.

Modular Lattices in $\mathcal{K}$ . . .

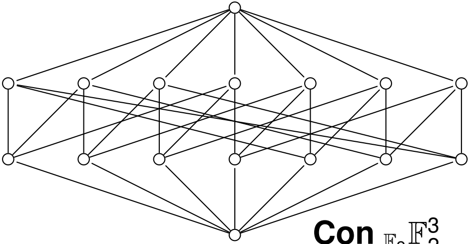
All modular lattices drawn above are **isometrically** embedded.

$\mathbf{M}_4$ is in $\mathcal{K}$ but cannot always be isometrically embedded.

Modular Lattices in $\mathcal{K}$ . . .

All modular lattices drawn above are **isometrically** embedded.

$\mathbf{M}_4$ is in $\mathcal{K}$ but cannot always be isometrically embedded.



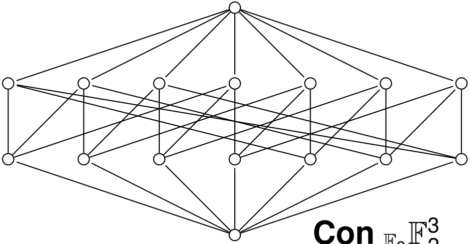
Con $_{\mathbb{F}_2}\mathbb{F}_2^3$

This lattice, the Fano plane, is not in $\mathcal{K}$.

Modular Lattices in $\mathcal{K}$ . . .

All modular lattices drawn above are **isometrically** embedded.

$\mathbf{M}_4$ is in $\mathcal{K}$ but cannot always be isometrically embedded.



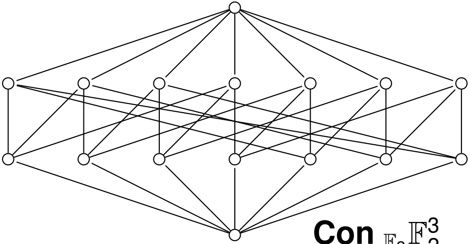
Con $_{\mathbb{F}_2}\mathbb{F}_2^3$

This lattice, the Fano plane, is not in $\mathcal{K}$. And it is possible some of you may have draw it.

Modular Lattices in $\mathcal{K}$ . . .

All modular lattices drawn above are **isometrically** embedded.

$\mathbf{M}_4$ is in $\mathcal{K}$ but cannot always be isometrically embedded.



Con ${}_{\mathbb{F}_2}\mathbb{F}_2^3$

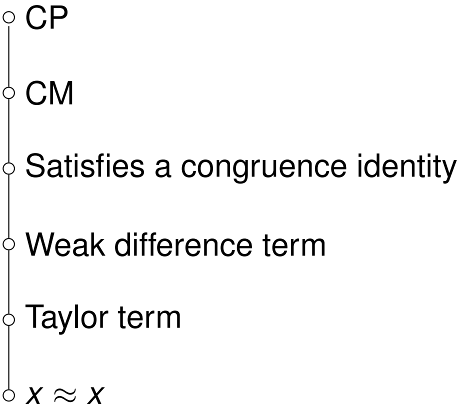
This lattice, the Fano plane, is not in $\mathcal{K}$. And it is possible some of you may have draw it.

Theorem

If $\mathbf{HS}\,\mathbf{Con}\,\mathcal{V}$ *contains a nondesarguean projective plane, it satisfies no congruence identity.*

Some Mal'tsev Classes of Varieties

In Walter's interpretability lattice:



Some Mal'tsev Classes of Varieties

Theorem

If $\mathcal{V}$ has a weak difference term and has an Abelian interval, there is a p, a prime or 0, so that $\mathbf{S}\,\mathbf{Con}\,\mathcal{V}$ contains $\mathcal{L}_p$, all subspace lattices of all finite dimensional vector spaces over the prime field of characteristic p.

Some Mal'tsev Classes of Varieties

Theorem

If $\mathcal{V}$ has a weak difference term and has an Abelian interval, there is a p, a prime or 0, so that $\mathbf{S}\,\mathbf{Con}\,\mathcal{V}$ contains $\mathcal{L}_p$, all subspace lattices of all finite dimensional vector spaces over the prime field of characteristic p.

Corollary

If $\mathcal{V}$ has a weak difference term and is not $CSD_\wedge$, then

$$\boxed{\mathcal{K} \subseteq \mathbf{S}\,\mathbf{Con}\,\mathcal{V}.}$$

Varieties with a Taylor Term

With a weak difference term, Abelian algebras are affine.
With a Taylor term they are (only) quasi-affine.

Varieties with a Taylor Term

With a weak difference term, Abelian algebras are affine.
With a Taylor term they are (only) quasi-affine.
But we do have:

Corollary

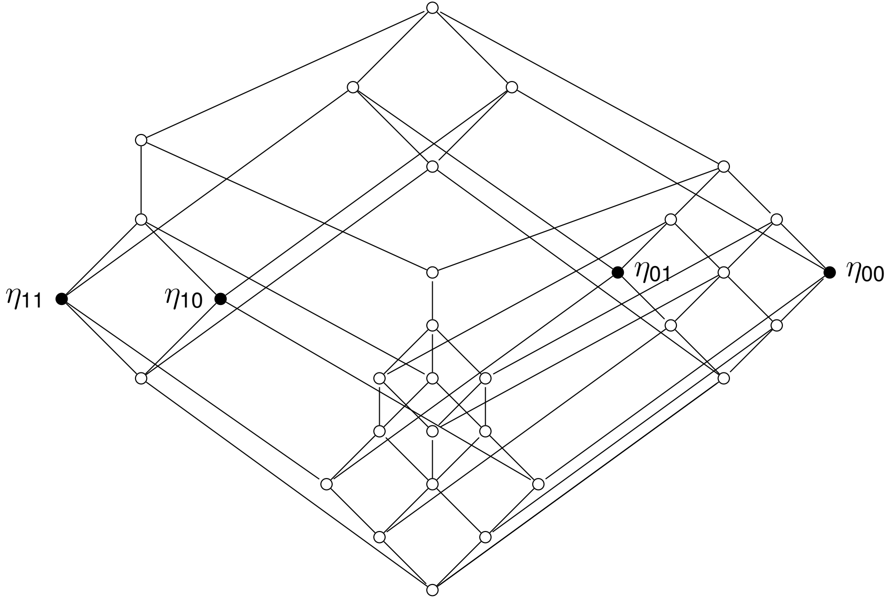
If $\mathcal{V}$ is idempotent, has a Taylor term term and is not CSD$_\wedge$, then

$$\boxed{\mathcal{K} \subseteq \boldsymbol{S}\,\mathbf{Con}\,\mathcal{V}.}$$

Problems

1. Without assuming a weak difference term, if there is a proper abelian interval (somewhere in $\mathcal{V}$), can we find an algebra in $\mathcal{V}$ with congruences $\theta \succ \varphi$ satisfying $C(\theta, \theta, \varphi)$?
2. Is there an abelian algebra with a Taylor term whose congruence lattice is a descending chain such that each proper image is not abelian?
3. (See next slide) $\mathbf{Con}\,\mathbf{F}_{\mathcal{P}}(1) \in \boldsymbol{HS}\,\mathbf{Con}\,\mathcal{V}$, whenever $\mathcal{V}$ is not CM. Is it always in $\boldsymbol{S}\,\mathbf{Con}\,\mathcal{V}$?

Problems . . .



References . . .

Emil Artin.
Coordinates in affine geometry.
Rep. Math. Colloquium (2), 2:15–20, 1940.

G. Birkhoff.
Lattice Theory.
Amer. Math. Soc., Providence, R. I., 1948.
rev. ed., Colloquium Publications.

P. Crawley and R. P. Dilworth.
Algebraic Theory of Lattices.
Prentice-Hall, Englewood Cliffs, New Jersey, 1973.

Alan Day and Ralph Freese.
A characterization of identities implying congruence modularity. I.
Canad. J. Math., 32(5):1140–1167, 1980.

R. Freese.
Finitely based modular congruence varieties are distributive.
Algebra Universalis, 32(1):104–114, 1994.

Ralph Freese, Christian Herrmann, and András P. Huhn.
On some identities valid in modular congruence varieties.
Algebra Universalis, 12(3):322–334, 1981.

Ralph Freese and Ralph McKenzie.
Commutator theory for congruence modular varieties, volume 125 of *London Mathematical Society Lecture Note Series*.
Cambridge University Press, Cambridge, 1987.
Online version available at: `http://www.math.hawaii.edu/~ralph/papers.html`.

References . . .

Ralph Freese and J. B. Nation.
Congruence lattices of semilattices.
Pacific J. Math., 49:51–58, 1973.

O. Frink.
Complemented modular lattices and projective spaces of infinite dimension.
Trans. Amer. Math. Soc., 60:452–467, 1946.

H. P. Gumm.
Geometrical Methods in Congruence Modular Algebras.
1983.
Memoirs Amer. Math. Soc.

C. Herrmann.
Affine algebras in congruence modular varieties.
Acta Sci. Math. (Szeged), 41:119–125, 1979.

David Hobby and Ralph McKenzie.
The structure of finite algebras, volume 76 of *Contemporary Mathematics*.
American Mathematical Society, Providence, RI, 1988.

B. Jónsson.
Representations of complemented modular lattices.
Trans. Amer. Math. Soc., 97:64–94, 1960.

B. Jónsson and G. S. Monk.
Representation of primary arguesian lattices.
Pacific J. Math., 30:95–139, 1969.

References . . .

Keith A. Kearnes and Emil W. Kiss.
The shape of congruence lattices.
Mem. Amer. Math. Soc., 222(1046):viii+169, 2013.

Keith A. Kearnes and Ágnes Szendrei.
The relationship between two commutators.
Internat. J. Algebra Comput., 8(4):497–531, 1998.

Paolo Lipparini.
Commutator theory without join-distributivity.
Trans. Amer. Math. Soc., 346(1):177–202, 1994.

S. V. Polin.
Identities in congruence lattices of universal algebras.
Mat. Zametki, 22(3):443–451, 1977.