Linear Mal'tsev Conditons

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General Algebra and its Applications


Coca-Cola

Mal'tsev Conditions

- (A. I. Mal'tsev) A variety $\mathcal{V}$ is CP iff there is a term $p(x, y, z)$ in the signature of $\mathcal{V}$ satisfying

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- (B. Jónsson) A variety is CD iff for some n it has terms $d_i(x, y, z)$ satisfying

$$\begin{aligned} &d_0(x, y, z) \approx x; \\ &d_i(x, y, x) \approx x && \text{for all } i; \\ &d_i(x, x, z) \approx d_{i+1}(x, x, z) && \text{if } i \text{ is even}; \\ &d_i(x, z, z) \approx d_{i+1}(x, z, z) && \text{if } i \text{ is odd}; \\ &d_n(x, y, z) \approx z. \end{aligned} \qquad (\Delta_n)$$

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- CM
- HM term
- TT
- $\mathrm{CSD}_\wedge$
- CSD
- k-permutable, for some k
- NU term
- wNU term
- cube term
- (weak) difference term

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- But if Σ is linear then the question is decidable.
- In fact:

Linear Good, non-Linear Bad

Theorem

Each of the following problems is decidable: for a finite set Σ of idempotent, linear equations, determine if $\mathcal{V}_\Sigma$ is

- CM
- HM
- *n-permutable, for some n*
- $\mathrm{CSD}_\wedge$
- CSD
- CD

$\mathcal{V}$ is **CR** (congruence regular) if $\theta \in \mathbf{Con}(\mathbf{A})$, $\mathbf{A} \in \mathcal{V}$ has a one-element block, then $\theta = 0_{\mathbf{A}}$. (Uniform congruences $\Rightarrow$ CR.)

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A variety $\mathcal{V}$ is congruence regular if and only if there exist ternary terms $g_1, \ldots, g_n$ and 4-ary terms $f_1, \ldots, f_n$ such that the following equations hold identically in $\mathcal{V}$.

$$\begin{aligned} g_i(x,x,z) &\approx z \qquad \text{for } 1 \leq i \leq n \\ x &\approx f_1(x,y,z,z) \\ f_1(x,y,z,g_1(x,y,z)) &\approx f_2(x,y,z,z) \\ &\vdots \\ f_n(x,y,z,g_n(x,y,z)) &\approx y \end{aligned}$$

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Is CR defined by a linear Mal'tsev condition?

Proving non-Linearity

Let A and B be **sets** and let

$$f : B \twoheadrightarrow A \text{ and } g : A \rightarrowtail B \text{ with } f(g(a)) = a.$$

So A is a set retraction of B via f and g.

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Let p be an n-ary operation on B. Define an n-ary operation $p_{f,g}$ on A by

$$p_{f,g}(a_1, \ldots, a_n) = f(p(g(a_1), \ldots, g(a_n))) \qquad (*)$$

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Theorem (W. Taylor)

An equational theory Σ has a linear basis iff its variety $\mathcal{V}$ is closed under basic set-retracts.

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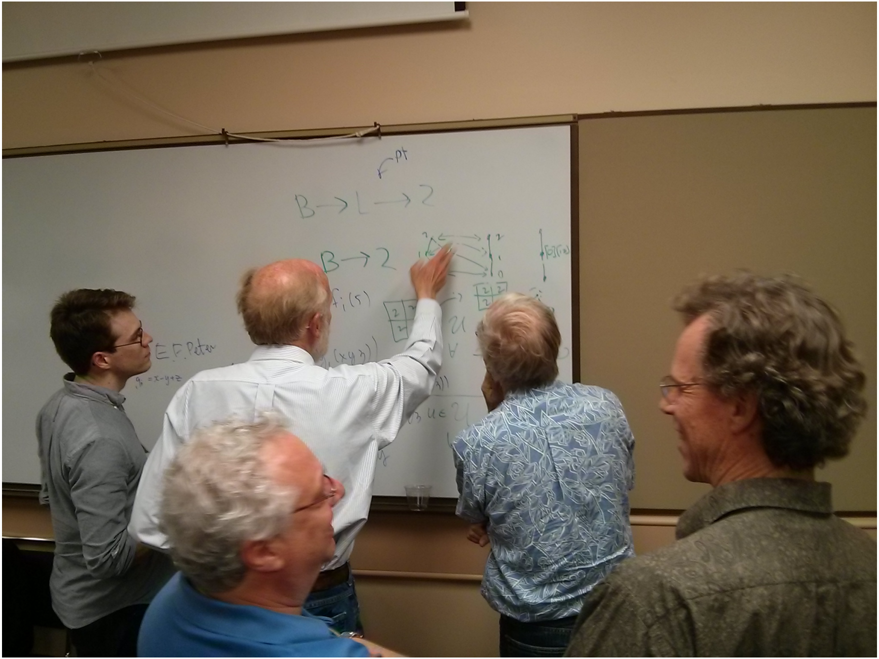
- Here's proof: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Theorem (Taylor, Kearnes, Sequeira, Szendrei)

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Proof.

By example: if **B** has a Mal'tsev term $p(x, y, z)$ then $\bar{p}^{\mathbf{A}}(x, y, z)$ is a basic operation of **A** which, by $(*)$ is a Mal'tsev term. □

Example: Having a semilattice term

A binary term $t(x, y)$ is called a **semilattice term** if

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Idea: Find **B** with a semilattice term and a full set retract **A** that doesn't.

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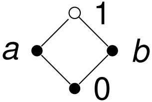

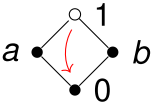

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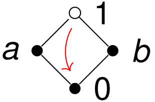

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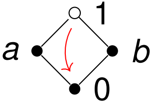

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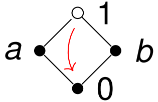
- If $a_1, \ldots, a_n \in A$, then the basic operations of $\mathbf{A}$ are

$$\bar{t}_n^{\mathbf{A}}(a_1, \ldots, a_n) = f(\bigvee a_i) = \begin{cases} 0 & \text{if } \bigvee a_i = 1 \\ \bigvee a_i & \text{otherwise} \end{cases}$$



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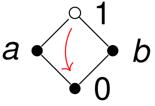

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Goal: Show $\mathbf{A} = \langle A, \bar{t}_2^{\mathbf{A}}, \bar{t}_3^{\mathbf{A}}, \ldots \rangle$ does not have a semilattice term.

Calculations with UACalc

Start with just $\bar{t}_2^{\mathbf{A}}$.

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Then $\mathbf{F}_{\mathbf{V}(\mathbf{A}_2)}(x, y)$ has only 5 elements:

$$x, \quad y, \quad \bar{t}_2(x, y), \quad \bar{t}_2(x, \bar{t}_2(x, y)) \quad \text{and} \quad \bar{t}_2(y, \bar{t}_2(x, y)).$$

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If we add $\bar{t}_3^{\mathbf{A}}$, $\mathbf{F}_{\mathbf{V}(\mathbf{A}_3)}(x, y)$ still only has 5 elements, suggesting $\mathbf{F}_{\mathbf{V}(\mathbf{A})}(x, y)$ has only 5 elements.

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Sidelight:

$|\mathbf{F}_{\boldsymbol{V}(\mathbf{A}_2)}(x, y, z)| = 96$, but $|\mathbf{F}_{\boldsymbol{V}(\mathbf{A}_3)}(x, y, z)| = 97$

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- so to show $\mathbf{F}_{\boldsymbol{V}(\mathbf{A})}(x, y) =$

$$\{x,\ y,\ \bar{t}_2(x, y),\ \bar{t}_2(x, \bar{t}_2(x, y)),\ \bar{t}_2(y, \bar{t}_2(x, y))\}$$

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- so to show $\mathbf{F}_{V(\mathbf{A})}(x, y) =$

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We just need to show it is closed under $\bar{t}_i$, $i = 2, 3, 4, 5$.

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Corollary

- *Having a semilattice term is not definable by a linear MC.*
- *Having a binary idempotent, commutative term with $t(x, t(x, y)) \approx t(x, y)$ is not definable by a linear MC.*

Digression

- (Kozik, Krokhin, Valeriote, Willard, Maroti, Janko)
  A finitely generated $\mathcal{V}$ is $\mathrm{CSD}_\wedge$ iff it has terms r, s and t with s a weak NU,

$$r(y, x, x) \approx t(y, y, x),$$
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- $\boldsymbol{V}(\mathbf{B})$ is $\text{CSD}_\wedge$ with $r = s = t = x \vee y \vee z = t_3$.
- So $\boldsymbol{V}(\mathbf{A})$ is $\text{CSD}_\wedge$ with $r = s = t = \bar{t}_3$.
- Question: does $\boldsymbol{V}(\mathbf{A}_2)$ satisfy $\text{CSD}_\wedge$?
- Yes (using UACalc)

$$r(x,y,z) = \bar{t}_2(x, \bar{t}_2(y,z))$$
$$s(x,y,z) = \bar{t}_2(\bar{t}_2(x,y), \bar{t}_2(z, \bar{t}_2(x,y)))$$
$$t(x,y,z) = \bar{t}_2(z, \bar{t}_2(x,y))$$

Example: Regularity is not Linear

- Congruence regularity cannot be defined by a linear Mal'tsev condition.
- Proof sketch:
  - Let $B = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$ with the 3-place operation $x + y + z$ modulo 2.
  - As **B** is the idempotent reduct of a vector space, the variety generated by **B** is congruence regular.
  - Let $A = \{0, 1, 2\}$ and define maps $f : B \to A$ and $g : A \to B$ by $f(\langle x, y\rangle) = x + y$ (so $f(\langle 1, 1\rangle) = 2$) and $g(0) = \langle 0, 0\rangle$, $g(1) = \langle 1, 0\rangle$ and $g(2) = \langle 1, 1\rangle$.
  - $|0, 2|1|$ is a congruence of **A**.

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- (W. Taylor) The class of varieties with no two element algebra form a Mal'tsev class. W. Taylor (1973).
- Example: the variety of groups of exponent 3.
- This class cannot be defined by a linear MC.

Origins and Motivation: TCT and Localization

- If A is a set retract of B, then $g(A) \subseteq B$ and so we can identify A with $g(A)$ and view A as subset of B.

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- If A is a set retract of B, then $g(A) \subseteq B$ and so we can identify A with $g(A)$ and view A as subset of B.
- Letting $e := g \circ f : B \to B$ we get a retraction map: $e(e(x)) = e(x)$.
- So $A = e(B)$ is a set retract of B.
- We make A into an algebra of signature $\bar{\sigma}$ by letting

$$\bar{t}^{\mathbf{A}}(a_1, \dots, a_n) = et^{\mathbf{B}}(a_1, \dots, a_n)$$


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