Projective Lattices
with applications to isotope maps and databases

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A lattice $\mathbf{L}$ is *projective* in a variety $\mathcal{V}$ of lattices if whenever

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is a *retraction* of $\mathbf{K}$: ρ is an endomorphism of $\mathbf{K}$ and $\rho^2 = \rho$. The sublattice $\rho(\mathbf{K}) = g(\mathbf{L})$ of $\mathbf{K}$ is isomorphic to $\mathbf{L}$. $g(\mathbf{L})$ is a *retract* of $\mathbf{K}$. In a slight abuse of terminology, we will also say $\mathbf{L}$ is a retract of $\mathbf{K}$. A lattice is projective in $\mathcal{V}$ if and only if it a retract of a free lattice $\mathbf{F}_{\mathcal{V}}(X)$ in $\mathcal{V}$.

Projective Lattices

Theorem (1978)

A lattice **L** *is projective (in the class of all lattices) iff*

- **L** *satisfies Whitman's condition* (*W*)*,*
- $L = \mathrm{D}(\mathbf{L}) = \mathrm{D}^d(\mathbf{L})$,
- **L** *has the minimal join cover refinement property and its dual, and*
- **L** *is finitely separable.*

Isotone Sections

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- Yes to all 3 if L (or P) is countable.

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    - where k is the order of σ.
- This theorem is also true for free Boolean algebras.

Finite Separability

- **L** is *finitely separable* if for each $a \in L$ there are finite sets
  - $\mathrm{A}(a) \subseteq \{x \in L : x \geq a\}$ and
  - $\mathrm{B}(a) \subseteq \{x \in L : x \leq a\}$ such that

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- (A for above and B for below.)
- Every countable lattice is finitely separable.

Finite Separability and Isotone Sections

Theorem

TFAE for a lattice $\mathbf{L}$

- $\mathbf{L}$ *is finitely separable.*
- *Every epimorphism* $\mathbf{K} \twoheadrightarrow \mathbf{L}$ *has an isotone section.*
- *An epimorphism* $f : \mathbf{FL}(X) \twoheadrightarrow \mathbf{L}$ *exists with an isotone section.*

An Interpolation Result

- $w \in \mathbf{F}_{\mathcal{V}}(X)$. There is a smallest subset $\mathrm{var}(w)$ of X with w in the sublattice generated by $\mathrm{var}(w)$.

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If $w \leq u$ in $\mathbf{F}_{\mathcal{V}}(X)$, there is a v with $w \leq v \leq u$ such that $\mathrm{var}(v) \subseteq \mathrm{var}(w) \cap \mathrm{var}(u)$ *and* $\mathrm{r}(v) \leq \min(\mathrm{r}(w), \mathrm{r}(u))$.

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Corollary

If $\mathbf{L}$ is a projective lattice in $\mathcal{V}$ then $\mathbf{L}$ is finitely separable.

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If **L** *is a projective lattice in* $\mathcal{V}$ *then* **L** *is finitely separable.*

Proof.

Finite separability is clearly preserved by retracts; thus it suffices to prove this theorem for $\mathbf{L} = \mathbf{F}_{\mathcal{V}}(X)$. For $a \in L$, let

$$\mathrm{A}(a) = \{w \in \mathbf{F}_{\mathcal{V}}(X) : w \geq a, \mathrm{var}(w) \subseteq \mathrm{var}(a), \text{ and } \mathrm{r}(w) \leq \mathrm{r}(a)\}$$

and define $\mathrm{B}(a)$ dually. The result follows from the interpolation theorem. □

Projective Ordinal Sums

Theorem

Let $\mathcal{V}$ be a variety of lattices and let $\mathbf{L} = \mathbf{L}_0 \dot{+} \mathbf{L}_1$, where $\mathbf{L}_i \in \mathcal{V}$ for $i = 0, 1$. Then $\mathbf{L}$ is projective in $\mathcal{V}$ if and only if both $\mathbf{L}_0$ and $\mathbf{L}_1$ are and one of the following hold:

1. $\mathbf{L}_0$ *has a greatest element.*
2. $\mathbf{L}_1$ *has a least element.*
3. $\mathbf{L}_0$ *has a countable cofinal chain and* $\mathbf{L}_1$ *has a countable coinitial chain.*

Projective Ordinal Sums

Corollary

$\mathbf{F}_{\mathcal{V}}(X) \dot{+} \mathbf{F}_{\mathcal{V}}(Y)$ *is projective in* $\mathcal{V}$ *iff*

- *X is finite,*
- *Y is finite, or*
- *both X and Y are countable.*

$\mathbf{F}_{\mathcal{V}}(\mathbf{P})$ for $\mathbf{P}$ a partially ordered set

Theorem

If $\mathbf{P}$ *is a partially ordered set, then* $\mathbf{F}_{\mathcal{V}}(\mathbf{P})$ (*the* $\mathcal{V}$ *lattice freely generated by* P *subject to the order relations of* $\mathbf{P}$) *is projective in* $\mathcal{V}$ *if and only if* $\mathbf{P}$ *is finitely separable.*

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- Example: $\mathbf{P}$ consists of two uncountable antichains A_0 and A_1 with $x_0 < x_1$ for $x_0 \in A_0$ and $x_1 \in A_1$.

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- Then $\mathbf{F}_{\mathcal{V}}(\mathbf{Q}) = \mathbf{F}_{\mathcal{V}}(A_0) + \mathbf{1} + \mathbf{F}_{\mathcal{V}}(A_1)$ is projective.

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- Then $\mathbf{F}_{\mathcal{V}}(\mathbf{Q}) = \mathbf{F}_{\mathcal{V}}(A_0) \dot{+} \mathbf{1} \dot{+} \mathbf{F}_{\mathcal{V}}(A_1)$ is projective.
- So $\mathbf{Q}$, and hence $\mathbf{P}$, is embeddable into a free $\mathcal{V}$ lattice.

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The (join) dependency relation D

- For join irreducibles $a \neq b$, a *depends* on b

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- $\mathrm{D}(\mathbf{L}) = L$ if the D relation is acyclic.

Applications of the dependency relation D

- These notions played a role in the study of
  - Free lattices
  - Projective lattices
  - Finitely presented lattices
  - Bounded homormorphisms
  - Transferable lattices
  - Congruence lattices of lattices
  - Representation of finite lattices as congruence lattices of finite algebras (Pudlák and Tůma).
  - Ordered direct bases in database theory.

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$$X_{k+1} = \begin{cases} X_k \cup \{b\} & \text{if } A \subseteq X_k \\ X_k & \text{otherwise} \end{cases}$$

The dependency relation in databases

- The *D-basis* of $\langle S, \phi \rangle$, Σ_D, is the union of

$$\{y \to x : y \neq x \in \phi(y)\} \text{ and } \quad (1)$$
$$\{X \to x : X \text{ a miminal cover of } x\}. \quad (2)$$

- *Ordered D-basis* is an ordering of Σ_D with (1) before (2).
- If Σ_D is ordered $\{s_1, s_2, \ldots, s_n\}$, then $\phi(X) = X_n$ where $X_0 = X$ and $s_{k+1} = A \to b$

$$X_{k+1} = \begin{cases} X_k \cup \{b\} & \text{if } A \subseteq X_k \\ X_k & \text{otherwise} \end{cases}$$

- So only one pass through Σ_D is needed to fine the closure.

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