Computing the TCT type set of an algebra

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References

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- J. Berman, E. Kiss, P. Pröhle, and Á. Szendrei, *The type set of a finitely generated variety*, Discrete Math. **112** (1993), 1–20.
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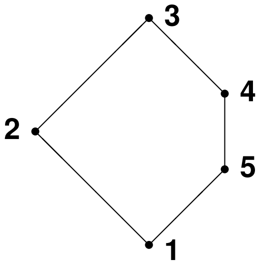
TCT



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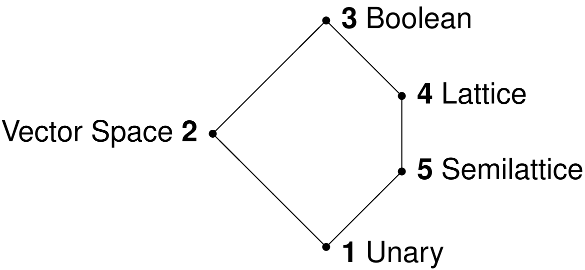

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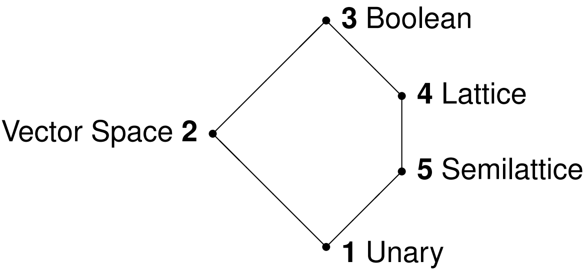

Theorem

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- The type set of **A** is determined by the **join irreducible** congruences, β (over their unique lower cover β_*).

Theorem

There is a linear time algorithm to compute $\mathrm{Cg}_{\mathbf{A}}(a, b)$.

Let $\mathbf{A} = \langle \mathbb{Z}_3, * \rangle^k$.

k	1	2	3	4	5
$\lvert \mathbf{A} \rvert$	3	9	27	81	243
$\#\mathrm{Cg}(a, b)$	2	13	68	337	1652
$\lvert \mathrm{J\,Con}\ \mathbf{A} \rvert$	2	9	31	97	291
$\lvert \mathrm{Con}\ \mathbf{A} \rvert$	3	34	4869	?	?

With $k = 4$, the closure gave 165000 elements after closing only 23 elements.

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Definition

$\{a, b\}$ is a **subtrace** if $\sigma(a, b) < \mathrm{Cg}(a, b)$.

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Lemma

- *If $\{a, b\}$ is a subtrace then $\{a, b\}$ is a minimal element of $\mathbf{G}_{a,b}$.*
- *If β is a join irreducible congruence on $\mathbf{A}$ then*
  - *$\{a, b\}$ is a β–subtrace if and only if it is a minimal element of $\mathbf{G}_\beta$.*

Finding a subtrace

$$\mathbf{S}_{a,b} := \{(f(a), f(b)) : f \in \mathrm{Pol}_1 \mathbf{A}\} = \mathrm{Sg}_{\mathbf{A}^2}(\{(a,b)\} \cup \Delta_2)$$

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In the end we get a pair $\{c,d\}$ such that all of $\mathbf{S}_{c,d}$ is generated. $\{c,d\}$ is a subtrace. We record if $(d,c) \in \mathbf{S}_{c,d}$. If it is, the type cannot be **4** or **5** and if it is not, the type cannot be **2** or **3**.

Finding the type

Assume $\{a, b\}$ is a subtrace of $\beta \succ 0$. Let

$$\mathbf{T}_{a,b} = \{(h(a,a), h(a,b), h(b,a), h(b,b)) : h \in \operatorname{Pol}_2 \mathbf{A}\}$$
$$= \operatorname{Sg}_{\mathbf{A}^4}(\{(a,a,b,b), (a,b,a,b)\} \cup \Delta_4)$$

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We may think of the elements of $\mathbf{T}_{a,b}$ as 2×2 tables, like

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are called a **join** and a **meet**.

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- If a join or a meet was found, the type is **5**.
- If a **one-snag** was found, the type is **2**.

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- If a join or a meet was found, the type is **5**.
- If a **one-snag** was found, the type is **2**.
- Otherwise the type is **1**.

Question

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Lemma

*If $\beta \succ 0$ is of type **4** or **5**, the transitve closure of $\mathbf{S}_{a,b}$ defines a compatible order, $\leq$, on $\mathbf{A}$.*

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*If $\beta \succ 0$ is of type **4** or **5**, the transitve closure of $\mathbf{S}_{a,b}$ defines a compatible order, $\leq$, on **A**.*

- *If $a \prec b$, then $\{a, b\}$ is a trace.*
- *If $\{a, b\}$ is a trace then $a < b$ or $b < a$.*

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Goal: Use the program to try to make examples with $|\mathbf{T}_{a,b}| = c|\mathbf{A}|^4$.

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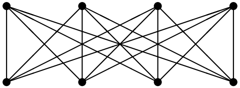

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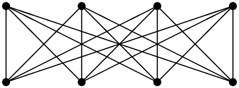

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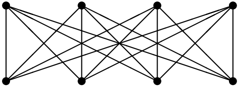

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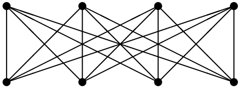

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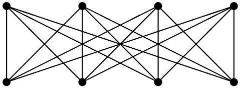

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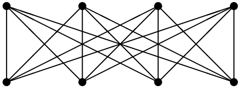

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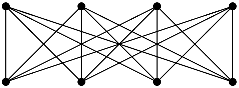

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- $|\mathbf{S}_{a,b}| = \binom{|\mathbf{A}|}{2}$.
- But the 4-tuples in $\mathbf{T}_{a,b}$ have at most 3 distinct elements.
- In fact $|\mathbf{T}_{a,b}| = |\mathbf{A}|^3/3 + |\mathbf{A}|^2/2 + |\mathbf{A}|/6$.

Suppose $(x, y, u, v) \in \mathbf{T}_{a,b}$. Then there is an $h \in \mathrm{Pol}_2(\mathbf{A})$ such that

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Type 5
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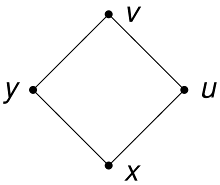
and so (x, y), (x, u), (y, v), (u, v) and (x, v) are all in $\mathbf{S}_{a,b}$ and thus traces unless they are in Δ_2.
Also

$$x \leq y \leq v \quad \text{and} \quad x \leq u \leq v$$

since this holds for the generators.



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Type **5**
More tries

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When $\mathbf{T}_{a,b}$ got big, the type changed from **5** to **4**.

Theorem

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Moreover, if

- *the type of* β *is* **5** *and*
- $(x, y, u, v) \in \mathbf{T}_{a,b}$,

then x *is a maximal element of* $\{z : z < y, z < u\}$.

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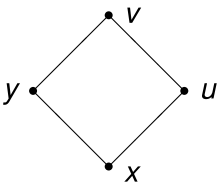
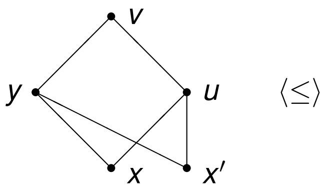
Theorem

Suppose $\beta \succ 0$, the type of β is **4** *or* **5***, $\{a, b\}$ is a trace for β and $\mathbf{T}_{a,b}$ contains a meet. Then the type of β is* **5** *if and only if $(x, y, u, v) \in \mathbf{T}_{a,b}$ implies x is the greatest lower bound of y and u in the $\sqsubseteq$ order.*

Type **5**
An example



Type 5
An example



Type **5**
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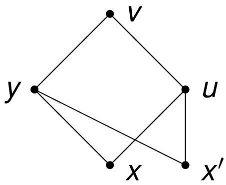
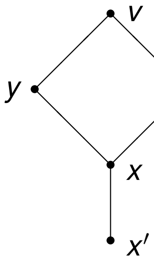


$\langle \leq \rangle$



$\langle \sqsubseteq \rangle$

Corollary

Suppose

- $\beta \succ 0$ *and*
- *the type of* β *is* **5** *and*
- $\{a, b\}$ *is a trace for* β*.*

Let $n = |\mathbf{A}|$*.*

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Moreover, for each $n \geq 2$ *there is a simple algebra of type* **5** *with* n *elements which achieves this bound.*

Corollary

Suppose that $\beta \succ 0$ *and that* $\{a, b\}$ *is a* $(0, \beta)$*–subtrace. Assume* $\mathbf{T}_{a,b}$ *has a meet and that* $(b, a) \notin \mathbf{S}_{a,b}$ *(so there is no involution and the type of* β *is either* **4** *or* **5***). If* $\mathbf{T}_{a,b}$ *contains*

	a	b
a	u	v
b	v	w

with $u \neq v$*, then the type of* β *is* **4***.*

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Theorem

If **A** *is a finite algebra,* $\beta \succ 0$ *in* **Con A** *and* $[\beta, \beta] = 0$*, then* $[\beta, \beta]_\ell = 0$*.*

Conclusion

Theorem

Suppose $\beta \succ 0$ *and the type of* β *is not* **3** *or* **4**. *Suppose* $\{a, b\}$ *is a subtrace for* β. *Then*

$$|\mathbf{T}_{a,b}(\mathbf{A})| \leq |\mathbf{A}|^3$$

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Corollary

The time to compute the type of a congruence $\beta \succ 0$ in an algebra **A** *is bounded by a constant times the cube of the input size (the size of the operation tables) of* **A**.