Maltsev products of CP varieties

Ralph Freese and Ralph McKenzie

k-Permutable Varieties

- A variety $\mathcal{V}$ has k-**permutable** congruences if $\forall\,\mathbf{A} \in \mathcal{V}$, and all θ and $\psi in\,$Con($\mathbf{A}$)

$$\theta \circ \psi \circ \theta \cdots = \psi \circ \theta \circ \psi \cdots$$

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- (А. И. Мальцев, 1954) $\mathcal{V}$ is congruence permutable iff it has a term $p(x, y, z)$ such that

$$p(x, z, z) \approx x, \quad p(x, x, z) \approx z.$$

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- For $\mathcal{U}$, $\mathcal{W}$ varieties,

$$\mathcal{U} \circ \mathcal{W} = \{\mathbf{A} : \exists\, \theta \in \mathrm{Con}(\mathbf{A}) \text{ with } \mathbf{A}/\theta \in \mathcal{W} \text{ and each block of } \theta \text{ in } \mathcal{U}\,\}$$

- If $\mathbf{B} \in \mathcal{U}$ and $\mathbf{C} \in \mathcal{W}$, then $\mathbf{B} \times \mathbf{C} \in \mathcal{U} \circ \mathcal{W}$. So

$$\mathcal{U} \vee \mathcal{W} \subseteq \boldsymbol{V}(\mathcal{U} \circ \mathcal{W}).$$

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- What about the Maltsev product of two CP varieties? Is it 3-permutable? Yes.

3-permutable

Theorem

Let **A** *be an idempotent algebra with congruence* θ *and terms* $p(x, y, z)$ *and* $q(x, y, z)$ *such that*

- *q is a Maltsev term for* $\mathbf{A}/\theta$ *and*
- *p is a Maltsev term for each* θ *block.*

Then **A** *has 3-permutable congruences.*

3-permutable

Lemma

Let **A** *satisfy the hypotheses of the theorem. Then* θ *3-permutes with every other congruence* ψ.

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Proof.

Suppose $(a, d) \in \theta \circ \psi \circ \theta$ so there exists b and c such that

$$a \;\theta\; b \;\psi\; c \;\theta\; d$$

We calculate

$$a = p(a, b, b) \;\psi\; p(a, b, c) \;\theta\; p(b, b, d) \;\psi\; p(c, c, d) = d$$

showing $(a, d) \in \psi \circ \theta \circ \psi$.

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Suppose $(a, d) \in \theta \circ \psi \circ \theta$ so there exists b and c such that

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showing $(a, d) \in \psi \circ \theta \circ \psi$.
Similarly $\psi \circ \theta \circ \psi \subseteq \theta \circ \psi \circ \theta$. □

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  - Varieties with a cube term are CM.

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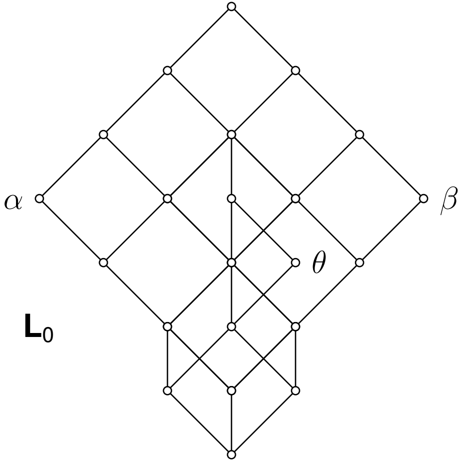
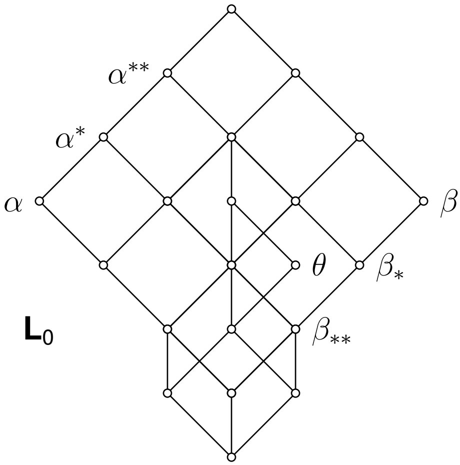
- So the sublattice **L** generated by α, β and θ is a homomorphic image of $\mathbf{L}_0$:

3-permutable



$\mathbf{L}_0$

3-permutable



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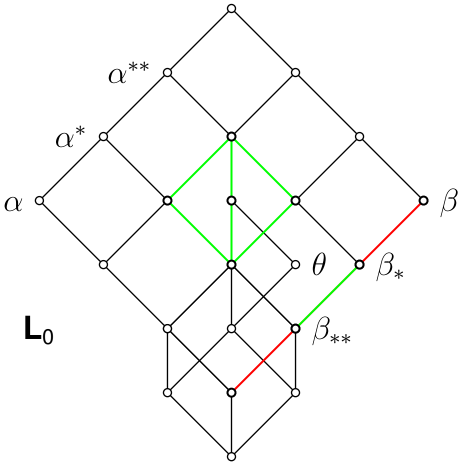

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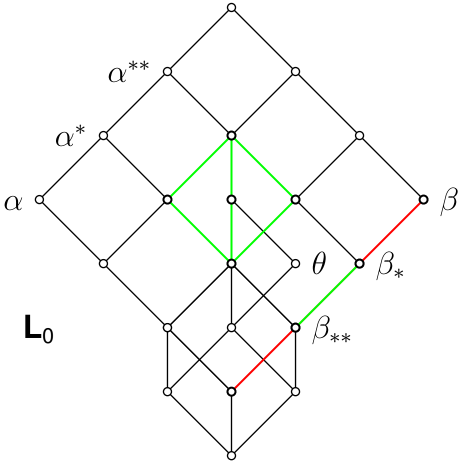

- $[\beta_*, \beta_*] \leq \alpha^*$ so β_* and α^* permute.

3-permutable

- Suppose

$$1 = \beta \circ \alpha^{**} \circ \beta \tag{1}$$

$$\alpha^{*} = \beta_{**} \circ \alpha \circ \beta_{**} \tag{2}$$

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$$1 = \beta \circ \alpha^{**} \circ \beta \quad (1)$$

$$\alpha^* = \beta_{**} \circ \alpha \circ \beta_{**} \quad (2)$$

- Then we calculate

$$\begin{aligned} 1 &= \beta \circ \alpha^{**} \circ \beta \\ &= \beta \circ \alpha^* \circ \beta_* \circ \beta \\ &= \beta \circ \alpha^* \circ \beta \\ &= \beta \circ \beta_{**} \circ \alpha \circ \beta_{**} \circ \beta \\ &= \beta \circ \alpha \circ \beta \end{aligned}$$

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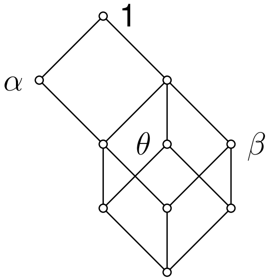
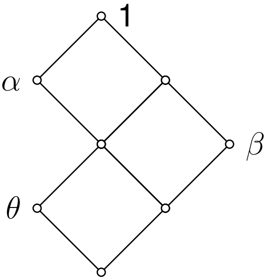
- A contradiction. So (1) or (2) must fail.

3-permutable

- If (1) fails, replace α by α^{**}. Then **L** is an image of $\mathbf{L}_2$.
- If (2) fails, replace β by β_{**} and **L** is an image of $\mathbf{L}_1$.



$\mathbf{L}_1$



$\mathbf{L}_2$

3-permutable

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3-permutable

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- In $\mathbf{L}_1$ let $\alpha_1 = \alpha \wedge \theta$ and $\beta_1 = \beta \wedge \theta$.

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- Suppose that $\mathbf{L}$ is an image of $\mathbf{L}_1$.
- In $\mathbf{L}_1$ let $\alpha_1 = \alpha \wedge \theta$ and $\beta_1 = \beta \wedge \theta$.
- By the above remarks $\theta = \alpha_1 \circ \beta_1 = \beta_1 \circ \alpha_1$.

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- Also θ 3-permutes with α

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$$\begin{aligned} 1 &= \theta \circ \alpha \circ \theta \\ &= \beta_1 \circ \alpha_1 \circ \alpha \circ \alpha_1 \circ \beta_1 \\ &= \beta_1 \circ \alpha \circ \beta_1 \\ &\subseteq \beta \circ \alpha \circ \beta \subseteq 1. \end{aligned}$$

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- Suppose that **L** is an image of $\mathbf{L}_1$.
- In $\mathbf{L}_1$ let $\alpha_1 = \alpha \wedge \theta$ and $\beta_1 = \beta \wedge \theta$.
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- Thus $\beta \circ \alpha \circ \beta = \alpha \vee \beta$, a contradiction.

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- Proof. In $\mathbf{L}_1$ let $\gamma = \beta \vee \theta$.

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- α and γ permute and θ and β 3-permute. So

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- Proof. In $\mathbf{L}_1$ let $\gamma = \beta \vee \theta$.
- α and γ permute and θ and β 3-permute. So

$$1 = \alpha \circ \gamma = \alpha \circ (\theta \circ \beta \circ \theta) = \alpha \circ \beta \circ \theta.$$

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- α and γ permute and θ and β 3-permute. So

$$1 = \alpha \circ \gamma = \alpha \circ (\theta \circ \beta \circ \theta) = \alpha \circ \beta \circ \theta.$$

- Let $a, d \in A$. Since $1 = (\alpha \circ \beta \circ \theta) \cap (\theta \circ \beta \circ \alpha)$ there are b, $c, e, f \in A$ with

$$a \; \alpha \; b \; \beta \; c \; \theta \; d \; \alpha \; e \; \beta \; f \; \theta \; a$$

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- Since p is Maltsev on the blocks of θ, we have

  $$p(d, c, b) \;\beta\; p(d, c, c) = d \quad \text{and} \quad p(e, e, a) \;\beta\; p(f, f, a) = a$$

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- $\theta \leq \alpha$, so $p(d, c, b) \; \alpha \; p(e, e, a)$. Hence

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- $\theta \leq \alpha$, so $p(d, c, b) \;\alpha\; p(e, e, a)$. Hence
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- Since a and d were arbitrary,

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- $\theta \leq \alpha$, so $p(d,c,b) \;\alpha\; p(e,e,a)$. Hence
  $$a \;\beta\; p(e,e,a) \;\alpha\; p(d,c,b) \;\beta\; d.$$
- Since a and d were arbitrary,
  $$1 = \alpha \vee \beta = \beta \circ \alpha \circ \beta,$$
- showing this case cannot occur and proving the theorem.

Rest of the Introduction

- (F-McK) If $q(x, y, z)$ is a Maltsev term for $\mathbf{A}/\theta$ and $p(x, y, z)$ is a Maltsev term for the blocks of θ, $\mathbf{A}$ has Hagemann-Mitschke terms $p_0 = x$, $p_4 = z$ and

$$p_1(x, y, z) = p(x, q(x, y, y), q(x, z, z))$$
$$p_2(x, y, z) = q(x, y, z)$$
$$p_3(x, y, z) = p(q(x, x, z), q(y, y, z), z)$$

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- (Valeriote) If $\mathcal{U}$ and $\mathcal{W}$ are CP varieties (of the same type) with Maltsev term p and q respectively then $\mathcal{U} \vee \mathcal{W}$ has Hagemann-Mitschke terms $p_0 = x$, $p_3 = z$ and p_1 and p_2 are

$$p(q(x, p(x, y, z), p(x, y, z)), q(x, p(y, z, z), z), q(x, y, z))$$
$$p(q(x, y, z), q(x, p(x, y, y), z), q(p(x, y, z), p(x, y, z), z))$$