Maltsev conditions

Ralph Freese
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Bill Lampe and J. B. Nation say Hi and wish Béla a Happy Birthday
Maltsev Conditions

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A variety $\mathcal{V}$ **realizes** $\Sigma$ if the function symbols occurring in $\Sigma$ can be interpreted as $\mathcal{V}$-terms such that the equations of $\Sigma$ hold.

(А. И. Мальцев, 1954) $\mathcal{V}$ is congruence permutable iff it realizes

$$\Sigma = \{p(x, y, y) \approx x, \ p(x, x, y) \approx y\}.$$ 

For groups

$$p(x, y, z) = xy^{-1}z$$

works.
Some Highlights:

- (early 1960’s) Pixley and Jónsson terms for CD.
Maltsev Conditions: A Recurring Theme

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Walter Taylor characterizes Maltsev classes.

Congruence varieties.

Tame Congruence Theory.

Kearnes-Kiss monograph.

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Universal Algebra Calculator

Ralph Freese

with help from

Emil Kiss, Matt Valeriote, Mike Behrisch, ...

Free at

www.uacalc.org

(Finds Maltsev, Jónsson, Hagemann-Mitschke, etc. terms.)
Definitions

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- If \( x \approx F(w) \), where \( w \) is a vector of not necessarily distinct variables, then \( F \) is \textbf{weakly independent} of its \( i^{th} \) place if \( w_i \neq x \).
Definitions

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- If $x \approx F(w)$, where $w$ is a vector of not necessarily distinct variables, then $F$ is **weakly independent** of its $i^{th}$ place if $w_i \neq x$. So

  \[ x \approx p(x, y, y) \]

  means $p(x, y, z)$ is weakly independent of its second and third place.
\( \Sigma' \), the \textbf{derivative} is the augmentation of \( \Sigma \) by equations that say that \( F \) is independent of its \( i^{th} \) place whenever \( \Sigma \) implies \( F \) is weakly independent of its \( i^{th} \) place.
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So if

\[
\Sigma = \{ p(x, y, y) \approx x, \ p(x, x, y) \approx y \}. 
\]

then \( p \) is weakly independent of all three places. So
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\[\Sigma = \{p(x, y, y) \approx x, \ p(x, x, y) \approx y\}.\]

Then \(p\) is weakly independent of all three places. So

\[\Sigma' \models x \approx p(x, x, x) \approx p(y, y, y) \approx y.\]

Thus \(\Sigma'\) is inconsistent.
If $\Sigma'$ is inconsistent then any variety that realizes $\Sigma$ is congruence modular (CM).

If $V$ is a CM variety, then $V$ realizes some $\Sigma$ such that $\Sigma'$ is inconsistent. (The Day terms work.)

The converse of the first statement is false: if $\Sigma$ is the lattice axioms, then $\Sigma' = \Sigma$. But the converse of the first statement is true if $\Sigma$ is linear (no nested composition in the terms occurring in $\Sigma$).

For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies CM.

This contrasts McNulty's Theorem that there is no effective way to decide if a (nonlinear) idempotent $\Sigma$ implies CM.
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A similar theorem holds for $\mathcal{V}$ satisfying some congruence identity if

"$\Sigma'$ is inconsistent"

is replaced by

"$\Sigma^{(k)}$ is inconsistent for some $k$."
Nice Consequences:
The Theorems of Dent, Kearnes, Szendrei

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- Day’s equations for CM have 3 variables:

$$m_i(x, y, y, z) \approx m_{i+1}(x, y, y, z), \quad \text{for } i \text{ odd}$$
Nice Consequences:

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- If replace this equation with

  $$m_i(x, y, y, y) \approx m_{i+1}(x, y, y, y), \quad \text{for } i \text{ odd}$$

  the resulting 2-variable system still defines modularity.
A **cube term** is one that is weakly independent of each of its variables.
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So a variety that has a cube term is CM.
The order derivative $\Sigma^+$: if

$$\Sigma \models x \approx F(w)$$

The we add

$$x \approx F(w')$$

to $\Sigma^+$, where, for each $i$, $w'_i = x$ or $w_i$. 
If some iterated order derivative $\Sigma^{+k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence $n$-permutable, for some $n$. 

If $V$ is a congruence $n$-permutable, for some $n$, then $V$ realizes some $\Sigma$ whose iterated order derivative $\Sigma^{+k}$ is inconsistent. (The Hagemann-Mitschke terms work.)

The converse of the first statement is false. But the converse of the first statement is true if $\Sigma$ is linear.

For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies congruence $n$-permutable, for some $n$. 

$n$-permutability

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Ralph Freese ()
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June 2012 12 / 1
Let $\Sigma_n$ be

\[ g_i(y, y, x) \approx x \quad 1 \leq i \leq n \]

\[ x \approx f_1(x, y, z, z, g_1(x, y, z)) \]

\[ f_1(x, y, z, g_1(x, y, z), z) \approx f_2(x, y, z, z, g_2(x, y, z)) \]

\[ f_2(x, y, z, g_2(x, y, z), z) \approx f_3(x, y, z, z, g_3(x, y, z)) \]

\[ \vdots \]

\[ f_n(x, y, z, g_n(x, y, z), z) \approx y \]
Let $\Sigma_n$ be

$$g_i(y, y, x) \approx x \quad 1 \leq i \leq n$$

$$x \approx f_1(x, y, z, z, g_1(x, y, z))$$

$$f_1(x, y, z, g_1(x, y, z), z) \approx f_2(x, y, z, z, g_2(x, y, z))$$

$$f_2(x, y, z, g_2(x, y, z), z) \approx f_3(x, y, z, z, g_3(x, y, z))$$

$$\vdots$$

$$f_n(x, y, z, g_n(x, y, z), z) \approx y$$

So $\Sigma_n^+ \models g_i(x, y, x) \approx x$. 
Let $\Sigma_n$ be

$$g_i(y, y, x) \approx x \quad 1 \leq i \leq n$$

$$x \approx f_1(x, y, z, z, g_1(x, y, z))$$

$$f_1(x, y, z, g_1(x, y, z), z) \approx f_2(x, y, z, z, g_2(x, y, z))$$

$$f_2(x, y, z, g_2(x, y, z), z) \approx f_3(x, y, z, z, g_3(x, y, z))$$

$$\vdots$$

$$f_n(x, y, z, g_n(x, y, z), z) \approx y$$

So $\Sigma_n^+ \models g_i(x, y, x) \approx x$.

Substituting $z \mapsto x$ now give $x \approx y$.  

A Maltsev Condition for Regular Varieties
Regular Varieties

Since the order derivative is weaker than the ordinary derivative, we get
Regular Varieties

- Since the order derivative is weaker than the ordinary derivative, we get

**Theorem (J. Hagemann)**

*Congruence regular varieties are*

- *Congruence n-permutable, for some n.*
- *Congruence modular.*
Σ⁺ inconsistent implies 3-permutable?

E. T. Schmidt constructed a regular variety that is not permutable. His example can be modified to give a k, but not k−1, permutable variety.
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E. T. Schmidt constructed a regular variety that is not permutable. His example can be modified to give a $k$, but not $k - 1$, permutable variety.
Semidistributivity

- The **weak derivative**, $\Sigma^*$, augments $\Sigma$ by an equation expressing that $F$ is independent of its $i^{th}$ place whenever

$$
\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)
$$

where the $y$ is in the $i^{th}$ place.
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**Theorem**

- If some iterated weak derivative $\Sigma^*^k$ of $\Sigma$ is inconsistent, then any variety that realizes $\Sigma$ is congruence semidistributive.
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**Theorem**

- If some iterated weak derivative $\Sigma^*_k$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence semidistributive.

- If $\mathcal{V}$ is a congruence semidistributive then $\mathcal{V}$ realizes some $\Sigma$ whose iterated weak derivative $\Sigma^*_k$ is inconsistent.
A new Maltsev Condition for Semidistributivity

\[ x \approx d_0(x, y, z) \]

\[ d_i(x, x, y) \approx d_{i+1}(x, x, y) \quad \text{for } i \equiv 0 \text{ or } 1 \mod 3 \]

\[ d_i(x, y, x) \approx d_{i+1}(x, y, x) \quad \text{for } i \equiv 0 \text{ or } 2 \mod 3 \]

\[ d_i(y, x, x) \approx d_{i+1}(y, x, x) \quad \text{for } i \equiv 1 \text{ or } 2 \mod 3 \]

\[ d_n(x, y, z) \approx z \]
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$$\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)$$

where the $y$ is in the $i^{th}$ place.

**Theorem**

- If some iterated weak derivative $\Sigma^{*k}$ of $\Sigma$ is inconsistent then any variety that realizes $\Sigma$ is congruence semidistributive.

- If $\forall$ is a congruence semidistributive then $\forall$ realizes some $\Sigma$ whose iterated weak derivative $\Sigma^{*k}$ is inconsistent.
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- **If $\forall$ is a congruence semidistributive then $\forall$ realizes some $\Sigma$ whose iterated weak derivative** $\Sigma^{*k}$ **is inconsistent.**

- The converse of the first statement is false, **even if $\Sigma$ is linear**. Nevertheless
Theorem

For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies congruence semidistributivity (or congruence meet semidistributivity).
Semidistributivity: Decidability

**Theorem**

For a finite linear, idempotent $\Sigma$ one can effectively decide if $\Sigma$ implies congruence semidistributivity (or congruence meet semidistributivity).

**Theorem (Kearnes, Kiss, Szendrei)**

Let $\Sigma$ be finite and idempotent. TFAE

- If $\forall$ realizes $\Sigma$ then $\forall$ is congruence meet semidistributive.
- $\Sigma$ is not realized in any variety of modules.
Example

\[ \Sigma = \{ f(x, x, x) \approx x, \ f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \} \]
Example

$$\Sigma = \{ f(x, x, x) \approx x, f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \}$$

For a ring $R$ does the variety of $R$-modules realize $\Sigma$?
Example

\[ \Sigma = \{ f(x, x, x) \approx x, \ f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \} \]

- For a ring \( \mathbf{R} \) does the variety of \( \mathbf{R} \)-modules realize \( \Sigma \)?
- If so \( f(x, y, z) = r_1 x + r_2 y + r_3 z \) for some \( r_i \in \mathbf{R} \).
Example

\[\Sigma = \{ f(x, x, x) \approx x, \ f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \}\]

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  Idempotency gives \( r_1 + r_2 + r_3 = 1 \).
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  Idempotency gives \( r_1 + r_2 + r_3 = 1 \). The other equations give \( r_1 = r_2 = r_3 \). So they are all 1/3.
Example

\[ \Sigma = \{ f(x, x, x) \approx x, \quad f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \} \]

- For a ring \( R \) does the variety of \( R \)-modules realize \( \Sigma \)?
- If so \( f(x, y, z) = r_1 x + r_2 y + r_3 z \) for some \( r_i \in R \).
  Idempotency gives \( r_1 + r_2 + r_3 = 1 \). The other equations give \( r_1 = r_2 = r_3 \). So they are all \( 1/3 \).
- So \( \Sigma \) is realized by \( R \)-modules iff 3 is invertible in \( R \).
Another Example

\[ \Sigma = \{ f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \approx f(x, x, x) \approx x \approx x \approx x \approx g(x, x, x, x) \approx x \approx x \approx g(x, x, x, y) \approx g(x, x, y, x) \approx g(x, y, x, x) \approx g(y, x, x, x) \approx g(x, x, y) \approx g(x, x, x, y) \} \]
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The variety of \( \mathbb{R} \)-modules realize \( \Sigma \)?
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- The variety of \( R \)-modules realize \( \Sigma \)?
- If so \( f(x, y, z) = r_1 x + r_2 y + r_3 z \) and
  \[ g(x, y, z, u) = s_1 x + s_2 y + s_3 z + s_4 u \] for some \( r_i \) and \( s_j \in R \).
Another Example

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\[ f(x, x, x) \approx x \quad g(x, x, x, x) \approx x \]  
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- If so \( f(x, y, z) = r_1 x + r_2 y + r_3 z \) and
  \( g(x, y, z, u) = s_1 x + s_2 y + s_3 z + s_4 u \) for some \( r_i \) and \( s_j \in \mathbb{R} \).
  As before this gives \( r_i = 1/3 \) and \( s_j = 1/4 \). The last equation gives \( r_1 = s_1 \).
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\[ \Sigma = \{ f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \]
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- The variety of \( R \)-modules realize \( \Sigma \)?

- If so \( f(x, y, z) = r_1 x + r_2 y + r_3 z \) and
  \[ g(x, y, z, u) = s_1 x + s_2 y + s_3 z + s_4 u \] for some \( r_i \) and \( s_j \in R \). As before this gives \( r_i = 1/3 \) and \( s_j = 1/4 \). The last equation gives \( r_1 = s_1 \).

- So \( 1/3 = 1/4 \) in \( R \). This implies \( 3 = 4 \), which gives \( 0 = 1 \). \( R \) is trivial.
Another Example

\[ \Sigma = \{ f(x, x, y) \approx f(x, y, x) \approx f(y, x, x) \] 
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- So \( 1/3 = 1/4 \) in \( R \). This implies \( 3 = 4 \), which gives \( 0 = 1 \). \( R \) is trivial.

- Hence any variety realizing \( \Sigma \) is congruence meet semidistributive.
Systems of linear equations like above can be put in the form

\[ AX = B \]

where \( A \) is an \( m \times n \) matrix over \( \mathbb{Z} \), \( B \) is a column vector over \( \mathbb{Z} \), and \( X \) is a column vector of ring variables.
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where $D$ is diagonal. It is easy to test if this has a solution.
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Corollary

For a finite idempotent linear \( \Sigma \), one can effective test if realizing \( \Sigma \) implies meet semidistributivity.
G. Czédli and G. Hutchinson used an analysis similar to the above, but much more detailed, in their characterization in terms of ring invariants of congruence varieties associated with varieties of modules in terms of ring invariants.
Semidistributivity

**Theorem (Kearnes, Kiss)**

A variety is congruence join semidistributive iff it is meet semidistributive and satisfies a nontrivial congruence identity.
Theorem (Kearnes, Kiss)

A variety is congruence join semidistributive iff it is meet semidistributive and satisfies a nontrivial congruence identity.

Corollary

For a finite idempotent linear $\Sigma$, one can effective test if realizing $\Sigma$ implies semidistributivity.
Decidable properties of finite, idempotent linear $\Sigma$’s

**Theorem**

For each property $P$ listed below, given a finite, idempotent, linear set of equations $\Sigma$ one can effectively decide if every variety that realizes $\Sigma$ satisfies $P$.

- Is congruence modular.
- Satisfies a nontrivial congruence identity.
- Is congruence $n$-permutable for some $n$.
- Is congruence semidistributive.
- Is congruence meet-semidistributive.
- Is congruence distributive.
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