Maltsev conditions

Ralph Freese

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Bill Lampe and J. B. Nation say Hi and wish Béla a Happy Birthday

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- A variety $\mathcal{V}$ **realizes** Σ if the function symbols occuring in Σ can be interpreted as $\mathcal{V}$-terms such that the equations of Σ hold.
- (А. И. Мальцев, 1954) $\mathcal{V}$ is congruence permutable iff it realizes

$$\Sigma = \{p(x, y, y) \approx x, \;\; p(x, x, y) \approx y\}.$$

- For groups

$$p(x, y, z) = xy^{-1}z$$

works.

Maltsev Conditions: A Recurring Theme

Some Highlights:

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- Kearnes-Kiss monograph.
- CSP.

Universal Algebra Calculator

Ralph Freese

with help from

Emil Kiss, Matt Valeriote, Mike Behrisch, ...

Free at

`www.uacalc.org`

(Finds Maltsev, Jónsson, Hagemann-Mitschke, etc. terms.)

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  $$x \approx p(x, y, y)$$

  means $p(x, y, z)$ is weakly independent of its second and third place.

The Derivative of Dent, Kearnes, Szendrei

- Σ', the **derivative** is the augmentation of Σ by equations that say that F is independent of its i^{th} place whenever Σ implies F is weakly independent of its i^{th} place.

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- So if
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then p is weakly independent of all three places. So
$$\Sigma' \models x \approx p(x,x,x) \approx p(y,y,y) \approx y.$$
Thus Σ' is inconsistent.

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- The converse of the first statement is false: if Σ is the lattice axioms, then $\Sigma' = \Sigma$. But

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- The converse of the first statement is true if Σ is linear (no nested composition in the terms occuring in Σ).
- For a finite linear, idempotent Σ one can effectively decide if Σ implies CM. This contrasts **McNulty's Theorem** that there is no effective way to decide if a (nonlinear) idempotent Σ implies CM.

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- A similar theorem holds for $\mathcal{V}$ satisfying some congruence identity if

  $$\text{"}\Sigma' \text{ is inconsistent"}$$

  is replaced by

  $$\text{"}\Sigma^{(k)} \text{ is inconsistent for some } k.\text{"}$$

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- If replace this equation with

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the resulting 2-variable system still defines modularity.

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- So a variety that has a cube term is CM.

Σ^+: A Derivative for n-permutability

- The **order derivative** Σ^+: if

$$\Sigma \models x \approx F(\mathbf{w})$$

The we add

$$x \approx F(\mathbf{w}')$$

to Σ^+, where, for each i, $\mathbf{w}'_i = x$ or $\mathbf{w}_i$.

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- The converse of the first statement is false. But
- The converse of the first statement is true if Σ is linear.
- For a finite linear, idempotent Σ one can effectively decide if Σ implies congruence n-permutability, for some n.

A Maltsev Condition for Regular Varieties

- Let Σ_n be

$$g_i(y,y,x) \approx x \qquad 1 \le i \le n$$

$$\begin{aligned} x &\approx f_1(x,y,z,z,g_1(x,y,z)) \\ f_1(x,y,z,g_1(x,y,z),z) &\approx f_2(x,y,z,z,g_2(x,y,z)) \\ f_2(x,y,z,g_2(x,y,z),z) &\approx f_3(x,y,z,z,g_3(x,y,z)) \\ &\vdots \\ f_n(x,y,z,g_n(x,y,z),z) &\approx y \end{aligned}$$

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f_1(x, y, z, g_1(x, y, z), z) &\approx f_2(x, y, z, z, g_2(x, y, z)) \\
f_2(x, y, z, g_2(x, y, z), z) &\approx f_3(x, y, z, z, g_3(x, y, z)) \\
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- So $\Sigma_n^+ \models g_i(x, y, x) \approx x$.
- Substituting $z \mapsto x$ now give $x \approx y$.

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Theorem (J. Hagemann)

Congruence regular varieties are

- *Congruence n-permutable, for some n.*
- *Congruence modular.*

Σ^+ inconsistent implies 3-permutable?

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- No:

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- No:
- E. T. Schmidt constructed a regular variety that is not permutable. His example can be modified to give a k, but not $k-1$, permutable variety.

Semidistributivity

- The **weak derivative**, Σ^*, augments Σ by an equation expressing that F is independent of its i^{th} place whenever

$$\Sigma \models x \approx F(x, \ldots, x, y, x, \ldots, x)$$

where the y is in the i^{th} place.

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A new Maltsev Condition for Semidistributivity

$$x \approx d_0(x, y, z)$$

$$d_i(x, x, y) \approx d_{i+1}(x, x, y) \quad \text{for } i \equiv 0 \text{ or } 1 \pmod 3$$

$$d_i(x, y, x) \approx d_{i+1}(x, y, x) \quad \text{for } i \equiv 0 \text{ or } 2 \pmod 3$$

$$d_i(y, x, x) \approx d_{i+1}(y, x, x) \quad \text{for } i \equiv 1 \text{ or } 2 \pmod 3$$

$$d_n(x, y, z) \approx z$$

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- The converse of the first statement is false, **even if Σ is linear**. Nevertheless

Semidistributivity: Decidability

Theorem

*For a finite linear, idempotent Σ one **can** effectively decide if Σ implies congruence semidistributivity (or congruence meet semistributivity).*

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Theorem (Kearnes, Kiss, Szendrei)

Let Σ be finite and idempotent. TFAE

- *If $\mathcal{V}$ realizes Σ then $\mathcal{V}$ is congruence meet semidistributive.*
- *Σ is not realized in any variety of modules.*

Example

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- If so $f(x,y,z) = r_1 x + r_2 y + r_3 z$ for **some** $r_i \in R$.

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- So Σ is realized by **R**-modules iff 3 is invertible in **R**.

Another Example

$$\Sigma = \{f(x,x,y) \approx f(x,y,x) \approx f(y,x,x)$$
$$f(x,x,x) \approx x \quad g(x,x,x,x) \approx x$$
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- So $1/3 = 1/4$ in **R**. This implies $3 = 4$, which gives $0 = 1$. **R** is trivial.
- Hence any variety realizing Σ is congruence meet semidistributive.

Systems of Linear Equations

- Systems of linear equations like above can be put in the form
$$AX = B$$
where A is an $m \times n$ matrix over $\mathbb{Z}$, B is a column vector over $\mathbb{Z}$, and X is a column vector of ring variables.

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Corollary

For a finite idempotent linear Σ, one can effective test if realizing Σ implies meet semidistributivity.

Czédli and Hutchinson

- G. Czédli and G. Hutchinson used an analysis similar to the above, but much more detailed, in their characterization in terms of ring invariants of congruence varieties associates with varieties of modules in terms of ring invariants.

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- *Is congruence distributive.*