Congruence Varieties

Ralph Freese

Hawaii

Conference on Lattice Theory
In honour of the 70th birthday of
George Grätzer
and
E. Tamás Schmidt

Congratulations!!

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Lemma

$\boldsymbol{P}\mathbf{Con}\,(\mathcal{K}) \subseteq \boldsymbol{S}\mathbf{Con}\,(\mathcal{K})$.
So $\boldsymbol{V}\mathbf{Con}\,(\mathcal{K}) \subseteq \boldsymbol{HS}\mathbf{Con}\,(\mathcal{K})$.

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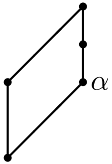
Conjecture (McKenzie, 1973)

There are no nonmodular congruence varieties other than the variety of all lattices.

An Amalgamation Technique

Suppose $\mathbf{N}_5 \leq \mathbf{Con}\,(\mathbf{A})$:



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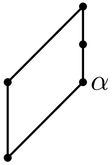

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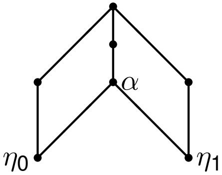

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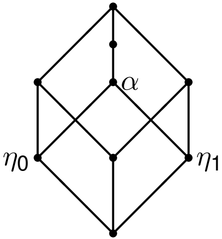

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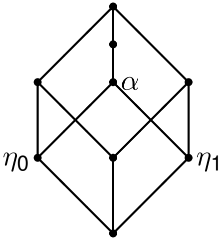

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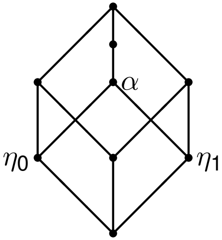

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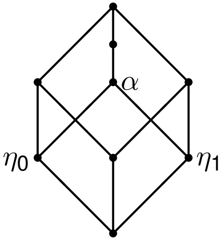

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Theorem (with G. Czédli)

- *There is an algorithm to decide is a lattice equation implies congruence modularity.*
- *And one to decide congruence distributivity.*

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Theorem (Pálfy, Szabó)

The congruence varieties of groups and of abelian groups are distinct.

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A few highlights:

- Does not assume the varieties are locally finite.
- Strong theory of solvability for varieties satisfying a congruence identity:
  - Congruences in solvable intervals permute.
  - Transposes of abelian (solvable) intervals are abelian (solvable).
  - **Con** (**A**) has a SD by modular decomposition.

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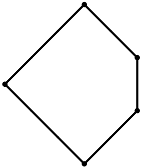

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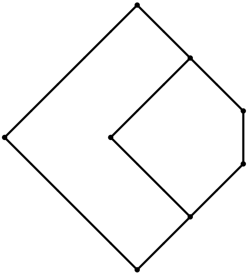

$\mathbf{N}_6$

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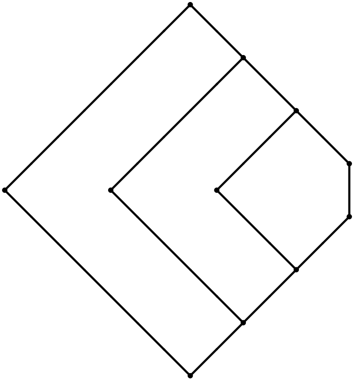

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Corollary

- *There is no largest, proper congruence variety.*

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- This and other results from the monograph and from Kearnes Szendrei give

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Let $\mathcal{K}$ be a variety. TFAE

- *$\mathcal{K}$ satisfies a nontrivial congruence identity.*
- $\mathbf{D}_2$ *is not a sublattice of a congruence lattice of a member of $\mathcal{K}$.*
- $\mathbf{D}_2$ *is not in the congruence variety of $\mathcal{K}$.*

An Application

Theorem

Let $\mathcal{K}$ is a variety with congruence variety $\mathcal{V}$. Suppose $\mathcal{V} \neq \mathcal{L}$ and $\mathbf{M}_3 \in \mathcal{V}$. Then $\mathbf{M}_3$ is projective in $\mathcal{V}$.

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- $\mathbf{F}_{\mathcal{V}}(3)$ is a sublattice of **Con** ($\mathbf{A}$), for some $\mathbf{A} \in \mathcal{K}$, (since ***SP*Con** $(\mathcal{K})$ = ***S*Con** $(\mathcal{K})$).

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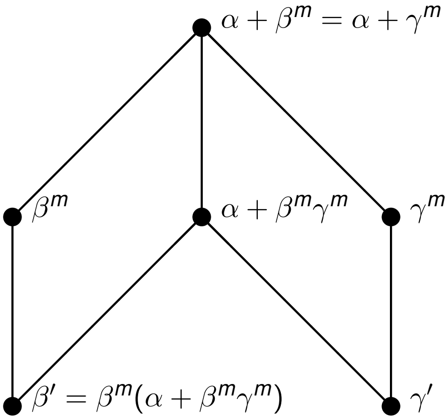
- A picture:

□

An Application



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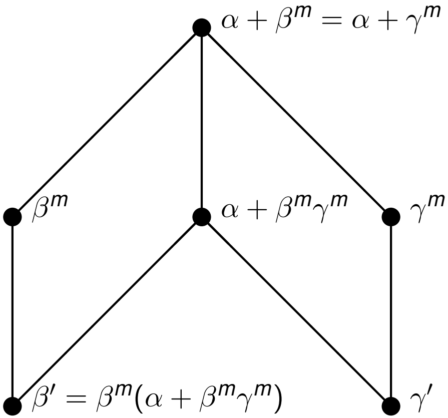

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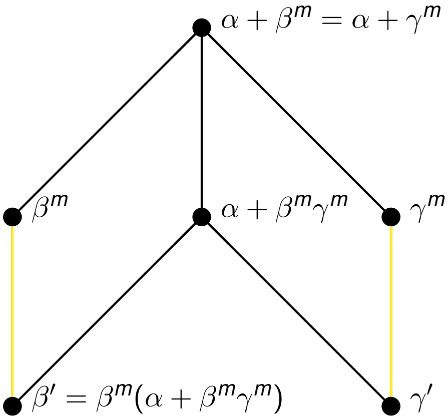
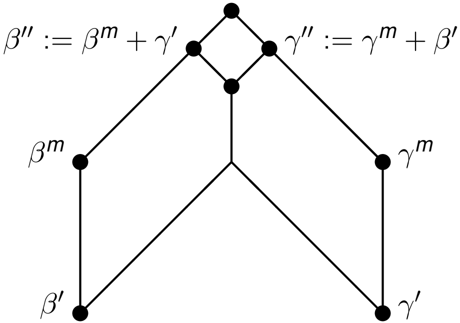


If $f(\alpha) = a$, $f(\beta) = b$ and $f(\gamma) = c$ is the homomorphism onto $\mathbf{M}_3$, then $f(\beta^m) = b$ and $f(\beta') = 0$. Also the interval $I[\beta', \beta^m]$ is abelian.

In the sublattice generated by β^m, β', γ^m, γ'



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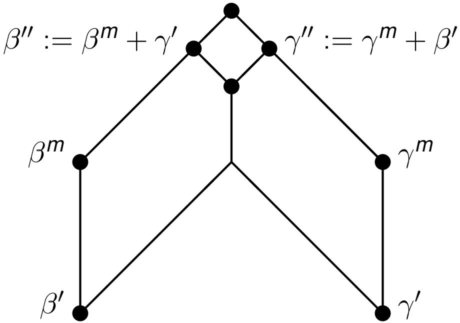

- If $\beta'' = \beta^m + \gamma'$ and $\gamma'' = \gamma^m + \beta'$, then the interval $I[\beta''\gamma'', \beta'' + \gamma'']$ is solvable and hence modular.

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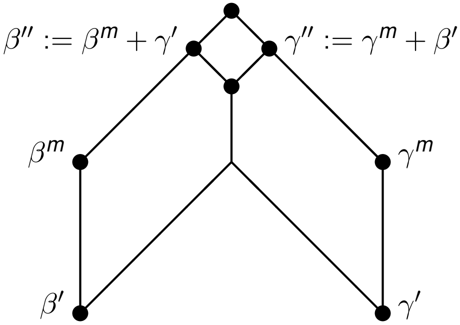

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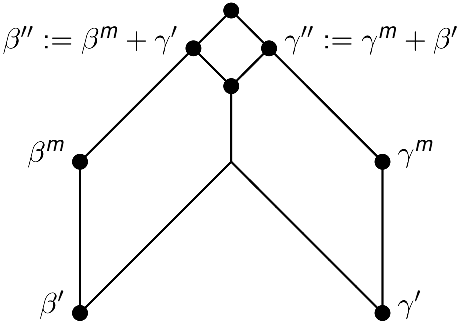

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- Letting $\alpha'' = \alpha(\beta'' + \gamma'') + \beta''\gamma''$ we get preimages of *a*, *b* and *c* in a modular interval.
- The result follows from the projectivity of $\mathbf{M}_3$ in modular lattices.

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- Recall the proof that lattices of permuting equivalence relations are arguesian; they satisfy

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1) \qquad (1)$$

where $c_0 = (a_1 + a_2)(b_1 + b_2)$, etc.

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- If these are (permuting) equivalence relations and (x, y) is in it, then

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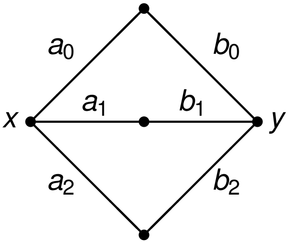

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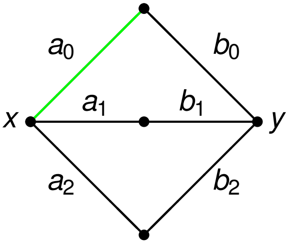

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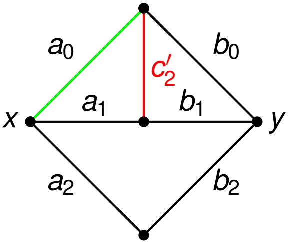

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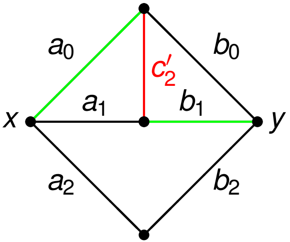

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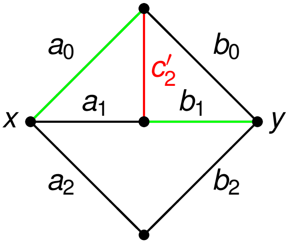

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Theorem

If a_i and b_i are elements of a lattice of equivalence relations and a_i permutes with b_i, $i = 0, 1$ and 2, then

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0 + b_1 + c_2(c_0 + c_1)$$

holds.

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- Applying f shows

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \le a_0 + b_1 + c_2(c_0 + c_1),$$

a contradiction.

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Theorem

Haiman's lattices, $\mathbf{H}_n$, lie in no proper congruence variety.

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