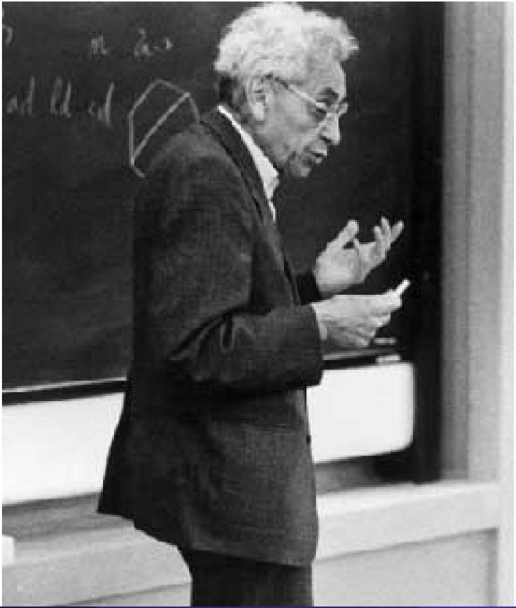
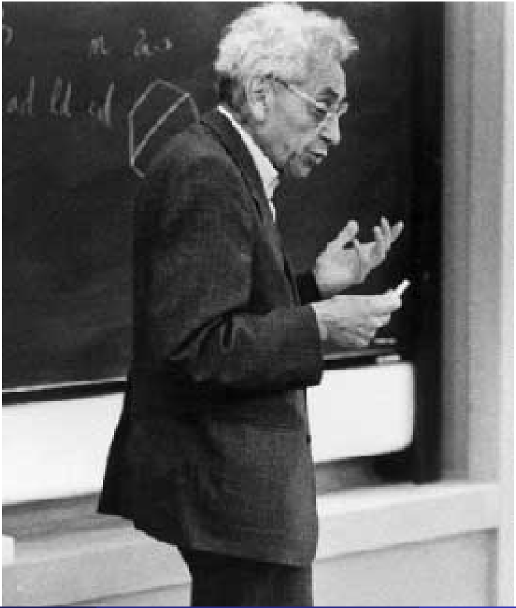
Sperner's Lemma

Ralph Freese

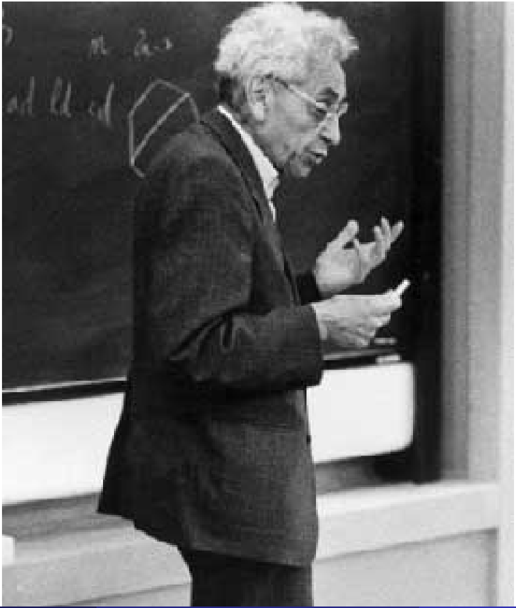
Paul Erdős



Pál Erdős



Erdős Pál



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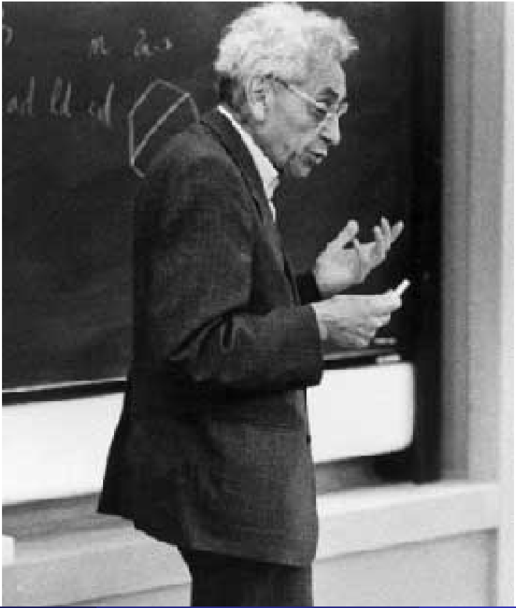

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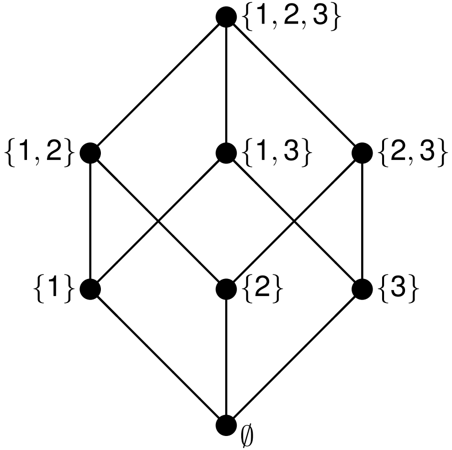

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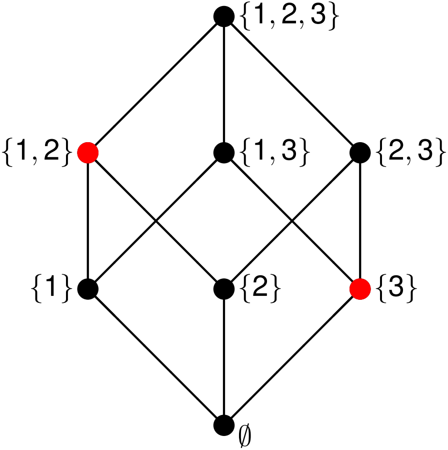

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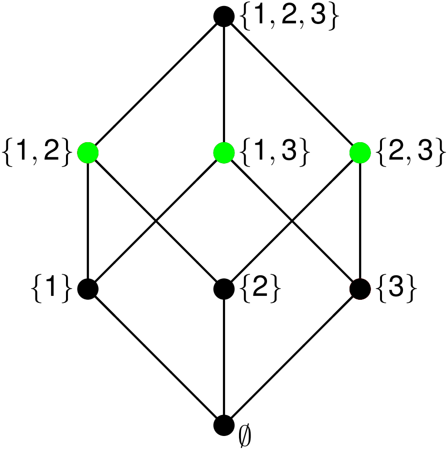

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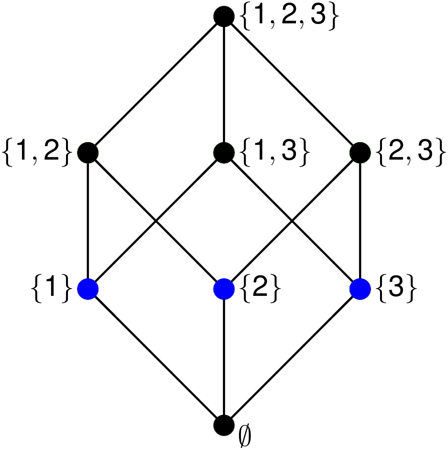

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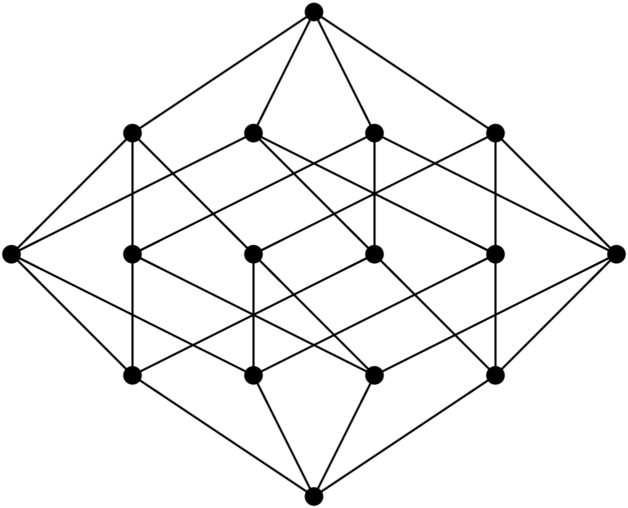

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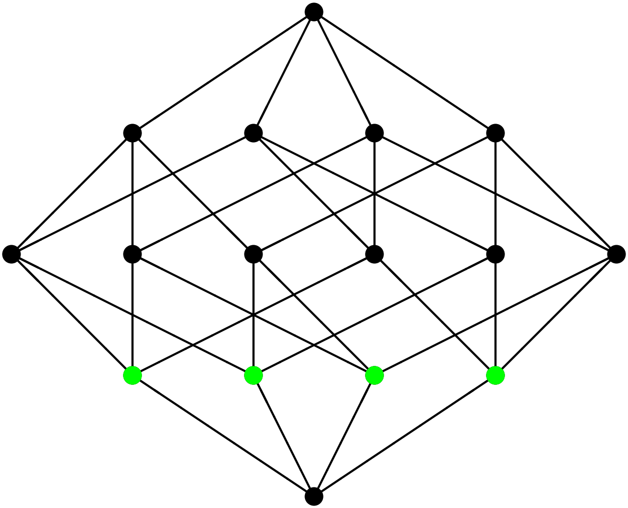

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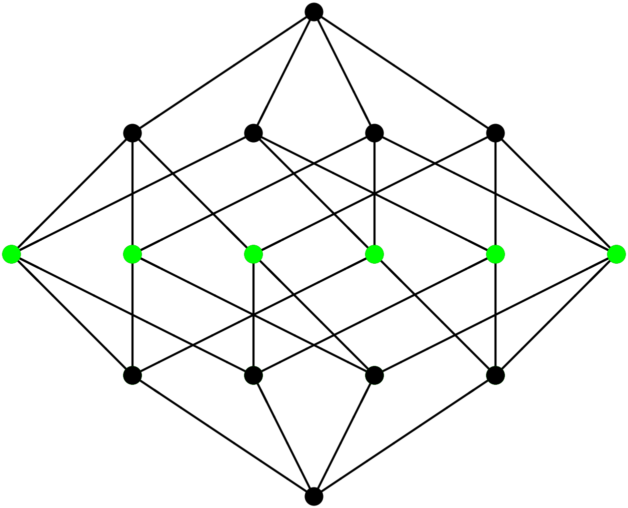

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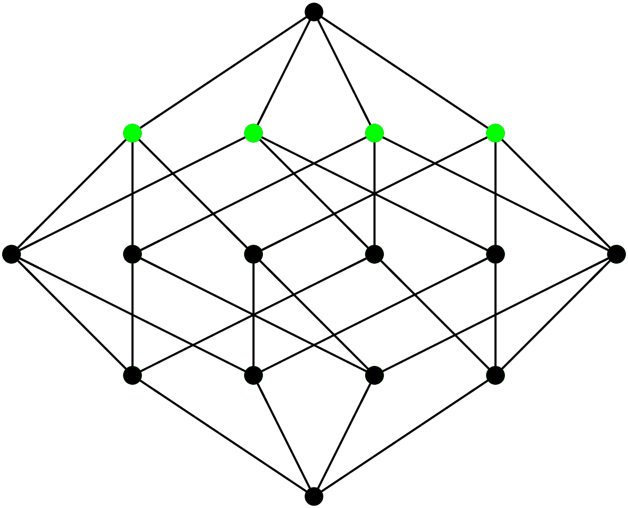

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Theorem (Sperner)

No!

Partially Ordered Sets and Lattices

- A set P together with a relation $\leq$ is called a **(partially) ordered set** if it satisfies:
  - $x \leq x$ (reflexive)
  - if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric)
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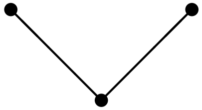

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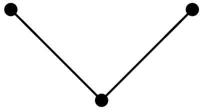

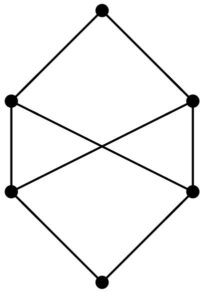

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Theorem (Kleitman, Edelberg, Lubell)

Yes!

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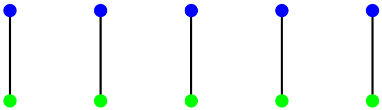

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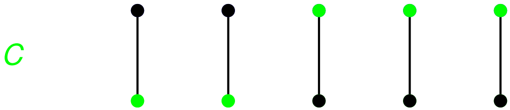

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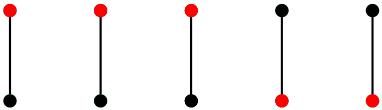

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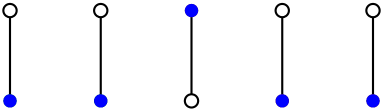

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- Since *L* is a finite lattice, it has a unique maximal (top) element, *T*.
- Clearly $\sigma(T) = T$, so the theorem of Kleitman, *et al.*, is true.

Proof of Dilworth's Theorem

- For A and $B \in L$, let

$$A_B = \{a \in A : a \geq b \text{ for some } b \in B\}$$
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- Since every element of P is comparable to something in A

$$A \vee B = A_B \cup B_A \quad \text{and} \quad A \wedge B = A^B \cup B^A$$
$$A_B \cup A^B = A \quad \text{and} \quad B_A \cup B^A = B$$
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- Using inclusion-exclusion:

$$\begin{aligned}
|A \vee B| + |A \wedge B| &= |A_B \cup B_A| + |A^B \cup B^A| \\
&= |A_B| + |B_A| - |A_B \cap B_A| + |A^B| + |B^A| - |A^B \cap B^A| \\
&= |A_B| + |A^B| + |B_A| + |B^A| - 2|A \cap B| \\
&= |A_B \cup A^B| + |A_B \cap A^B| + |B_A \cup B^A| + |B_A \cap B^A| - 2|A \cap B| \\
&= |A| + |B|
\end{aligned}$$

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Back to Sperner

- Let S be a set with n elements and let $P = \mathcal{P}(S)$.
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- So if $X \in \mathcal{A}$ has k elements, then $\mathcal{A}$ contains all subsets with k elements. This (easily) implies $\mathcal{A}$ must consist of all subsets of S of size k, for some k, which proves Sperner's Lemma.

Partition Lattices and Rota's Problem

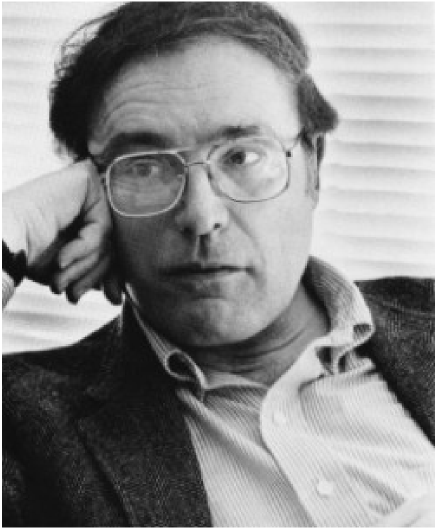
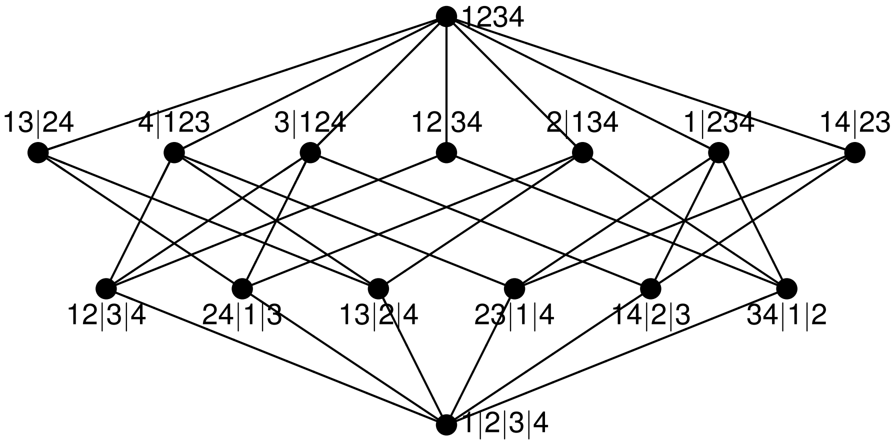


Figure: Gian-Carlo Rota

Partition Lattices and Rota's Problem



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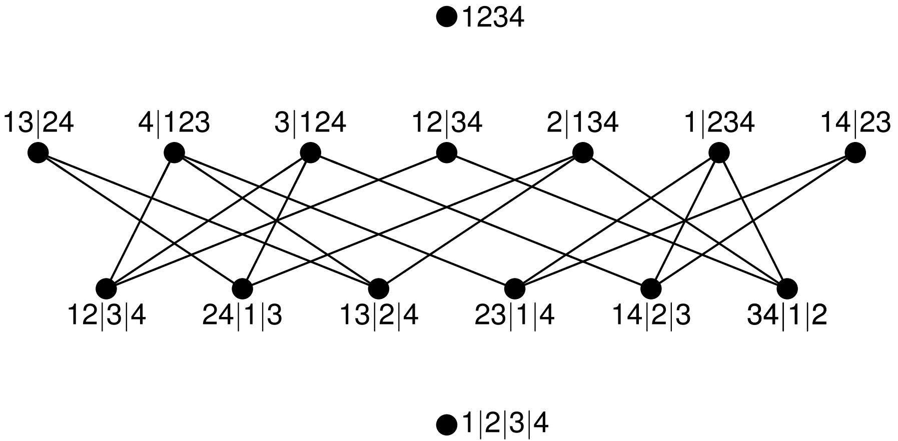

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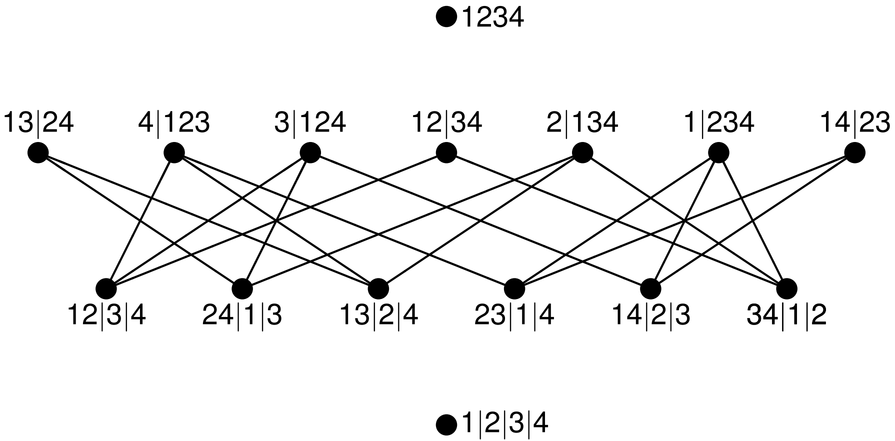

- **Gian-Carlo Rota:** Does the largest sized antichain in the lattice of partitions of an n element set consist of all partitions with k block, for some k.

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Problem

What is Dilworth's lattice for the lattice of equivalence relations of an n element set?