Probability Space: \((\Omega, \mathcal{F}, P)\)

\(\Omega\) somehow indexes all "primitive" or atom-like possibilities. It is a non-empty set.

\(\mathcal{F}\) is a collection of subsets of \(\Omega\), called **events**. It is required to be a \(\sigma\)-algebra.

\(P\) is a function giving each event \(A\) a probability, a number \(P(A)\) such that \(0 \leq P(A) \leq 1\). We require \(P(\Omega) = 1\). We also require **countable additivity**.

**Basic Example:** toss one fair coin. Let \(\Omega = \{H, T\}\).

\[
\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}.
\]

\(P(\emptyset) = 0\), \(P(\{H\}) = 1/2\), \(P(\{T\}) = 1/2\) and \(P(\{H, T\}) = 1\).
The concept here is that any (real) number between 0 and 1 is equally likely, measured according to length of intervals.

- $\Omega = [0, 1]$ (variously, $(0, 1)$, $[0, 1)$, etc.)
- $\mathcal{F}$ is $\mathcal{B}$, the Borel subsets of $\Omega$
- Here,

$$P((a, b]) = P((a, b)) = P([a, b)) = P([a, b]) = b - a$$

for $0 \leq a \leq b \leq 1$.

Matlab’s rand command simulates this. Note a “paradox”: $P\{x\} = 0$ for each $x$ but $P([0, 1]) = 1$. 
Here $\Omega = \mathbb{R}$, the set of all real numbers, and $\mathcal{F}$ is the set of Borel subsets of $\mathbb{R}$. For real numbers $a \leq b$

$$P([a, b]) = P([a, b)) = P((a, b]) = P((a, b)) = \int_{a}^{b} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \, dt$$

One proves that $P(\Omega) = 1$ using two-dimensional integrals and a change of variables to polar coordinates.

- In general, to compute $P$ requires approximation integration.

$P([-1, 1]) \approx 0.682689492137$, $P([-2, 2]) \approx 0.954499736104$ and $P([-3, 3]) \approx 0.997300203937$

- Bell shaped integrand, and symmetric with respect to 0

- Natural to any situation that is the result of many small, additive, independent influences with an overall bound on the variances of the small influences.
Def’n of a Probability Space

Third Continuous Example

See Example 1.1.4 on pages 4-7 in the textbook.

- $\Omega = \Omega_\infty$ consists of all infinite sequences of $T$s and/or $H$s. For $\omega \in \Omega$, let $\omega_j$ be the letter in the $j$-th position of $\omega$.
- $P(\emptyset) = 0$ and $P(\Omega) = 1$
- Let $p > 0$ such that $q = 1 - p > 0$ also. (This is equivalent to saying $0 < p < 1$.)
- Let $n$ be a positive integer, and let $\tau_1, \ldots, \tau_n$ be any choice of $T$ and/or $H$ for the values. Let

$$A = \{ \omega \in \Omega : \text{for } 1 \leq j \leq n, \omega_j = \tau_j \}$$

Then $P(A) = p^k q^{n-k}$ where $k$ is the number of $j \in [1, n]$ where $\tau_j = H$.

- Let $\mathcal{F} = \mathcal{F}_\infty$ be the $\sigma$-algebra generated by all sets $A$ that have the form described in the previous item.
If $A_j$, $1 \leq j < \infty$, is a sequence of disjoint events, then

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

Disjoint means that $A_j \cap A_k = \emptyset$ for $j \neq k$. 

$\sigma$-additive (countably additive) Definition
$\sigma$-additive (consequences of the idea)

- $P(\emptyset) = 0.$
- $P(A^c) = 1 - P(A)$.
- Additivity works for unions of finitely many disjoint events.
- If $A \subset B$, then $P(A) \leq P(B)$. (See Exercise 1.1 on page 41.)
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$. 
First Matlab Simulation

Ω is finite

% Input an outcome vector that encodes Ω, omega
% Input a vector of probabilities for each \{ω\}, pvec
% Make sure that pvec has no negative entries and sums to 1
function trial=randomdraw(omega,pvec)
n=length(omega); u=rand();
i=0;
runningsum=0;
while i<n && u>=runningsum
    i=i+1;
    runningSums=runningSums+pvec(i);
    trial=omega(i);
end
Imagine tossing a fair coin, with independent tosses, until you get a first H. Here

\[ \Omega = \{H, TH, TTH, TTTTH, TTTTTH, \ldots\} \cup \{TTT\ldots TTT\ldots\} \]

Here \( \mathcal{F} \) can be all the subsets of \( \Omega \) (a very large infinity of sets). The most natural probability assignment is that

\[ P(\{\omega\}) = 2^{-n} \quad \text{if } \omega \text{ has } n - 1 \text{ Ts followed by one H} \]

Because

\[ 1 = \sum_{n=1}^{\infty} 2^{-n}, \]

there is no probability left over for the primitive outcome \( \omega_\infty \) of all Ts. So \( P(\{\omega_\infty\}) = 0. \)
function trial=WaitingForH()
% The output is the position of the coin toss
% where there is a first H
% THIS PROGRAM IS FLAWED
u=rand();
i=0;
runningsum=0;
while u>=runningsum
    i=i+1;
    runningsum=runningsum+2 \cdot (\frac{1}{2})^{i};
    trial=i;
end
Experiments with Matlab

- Use the digits of your student ID as $\Omega$. Give each digit equal probability except when a digit repeats. If a digit repeats, say $k$ times, give it $k$ times the probability of any digit that occurs only once. List the distinct digits as a column, and next to each digit $s$ give $P(\{s\})$.

- Using the $\Omega$ and $P$ of the previous problem, run the first Matlab simulation 1000 times and save the answers (to a Matlab vector would be easiest). Tally how often each digit appears, and append a column of these tallies to your answer to the previous problem. Finally, add a fourth column with the differences $P(\{s\}) - c(s)/1000$ where $c(s)$ is the tally for $s$.

- Repeat Problem 2 but with 10,000 in the place of 1000.

- Repeat Problem 2 with 100,000 in the place of 1000.
For each of Problems 2, 3, and 4, compute the largest discrepancy in absolute value (you should have 3 of these answers). Call them \( d_3, d_4 \) and \( d_5 \), respectively. Find one negative exponent \( \alpha \) for \( 10^n \), for \( n = 3, 4 \) and \( 5 \) so that

\[
(10^n)^\alpha \approx d_n
\]

There is a serious flaw with the Second Matlab Simulation. When \( i \) becomes big enough, \( 2^{\wedge}(-i) \) when added to \( \text{runningsum} \) does not change \( \text{runningsum} \). What is the first \( i \) where this happens?

Fix the second Matlab simulation to stop incrementing \( i \) before \( 2^{\wedge}(-i) \) becomes too small to change \( \text{runningsum} \). Also, modify the output so that \( \text{trial} = \text{lasti} \) when (theoretically) \( i \) could continue past \( \text{lasti} \).
Pick a NYSE ticker symbol that begins with the same letter as your last name. Get closing daily prices for one year (say, the year 2008). Make these prices a long vector, which I’ll call $x$. It should have about 250 components or entries. In Matlab the $i$-th entry is accessed as $x(i)$. Make a new vector, $y$, which is one entry shorter than is $x$: $y(i) = \ln(x(i+1)) - \ln(x(i))$ (look up the Matlab command for \texttt{ln}).

Compute the mean $\bar{y}$ and sample standard deviation $s_y$ of the entries of $y$. $\bar{y}$ is just their sum divided by the number of entries (there is a Matlab command for it). The sample standard deviation is

$$s_y := \sqrt{\frac{\sum_{i=1}^{\text{length}(y)} (y(i) - \bar{y})^2}{\text{length}(y) - 1}}$$

There is also a Matlab command for this.
Let $z$ be the $y$-vector "normalized" to have have mean 0 and sample standard deviation 1: $z = (y - \bar{y})/s_y$ will do the trick in Matlab. Print out a Matlab histogram of the entries of $z$.

Plot the curve

$$\text{normalpdf}(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad \text{for } -5 < t < 5$$

Try to overlay the plot of the previous item with the histogram of $z$ on the same scale (Matlab may resist doing this, but the "hold on" command might help).
Make a table with eight rows. Put in column 1 the following bins: \((-\infty, -3], (-3, -2], (-2, -1], (-1, 0], (0, 1], (1, 2], (2, 3], (3, \infty)\). In column 2 put the count of how many entries of \(z\) are in the bin for that row. In column 3, put the relative frequency (count of \(z\)/total number of entries). In column 4, put the standard normal probability assigned to the bin for that row. In column 5 put the values from column 4 minus those of column 3, row by row.

Research and apply the \(\chi^2\) goodness of fit test to the table of data in the last problem. Can you conclude that the \(z\)-values are modeled well by a normal distribution?