An 1890 Example From H. A. Schwartz of How Not To Define Surface Area

I learned of this example from T. W. Körner's book, *A Companion to Analysis*, page 218.

The Surface. Let the surface be a cylinder of height 1 and radius 1. For concreteness, one can let the cylinder be parameterized as $\mathbf{r}(u, v) = \langle \cos u, \sin u, v\rangle$ for $0 \le v \le 1$ and $0 \le u \le 2\pi$.

The Triangulation. Let $u_i = 2\pi/n$ for $0 \le i \le n$ and $v_j = j/m$ for $0 \le j \le m$. Set $P_{i,j} = \mathbf{r}(u_i, v_j)$. Also, for $0 \le i < n$ and $0 \le j < m$, let $X_{i,j} = \mathbf{r}(u_i + \pi/n, v_j + 1/(2m))$. Note that

(1) The points $P_{i,j}$ are a rectangular grid upon the surface, with $P_{n,j} = P_{0,j}$.
(2) Each $X_{i,j}$ is the midpoint of a rectangle whose corners are $P_{i,j}$, $P_{i,j+1}$, $P_{i+1,j}$ and $P_{i+1,j+1}$.

With $0 \le i < n$ and $0 \le j < m$, the triangulation of the surface consists of mn sets of four flat, planar triangles with these vertices:

(1) A "top" triangle with vertices $P_{i,j+1}$, $P_{i+1,j+1}$ and $X_{i,j}$.
(2) A "right side" triangle with vertices $P_{i+1,j+1}$, $P_{i+1,j}$ and $X_{i,j}$.
(3) A "bottom" triangle with vertices $P_{i,j}$, $P_{i+1,j}$ and $X_{i,j}$.
(4) A "left side" triangle with vertices $P_{i,j}$, $P_{i,j+1}$ and $X_{i,j}$.

The basic idea is that the sum of the areas of these $4mn$ triangles should approximate the area of the cylinder which is 2π. This idea is badly wrong. Depending on the ratio of m to n as $m \to \infty$ and $n \to \infty$, the limit can fail to exist, be ∞ or be any finite number in $[2\pi, \infty)$.

Finding the area of any "left side" triangle or "right side" triangle. The gentle reader is asked to notice that any "right side" triangle is congruent to any "left side" triangle. Thus, we need compute the area of just one of these types, say a "left side" triangle.

The points $P_{i,j}$ and $P_{i,j+1}$ lie on a vertical line in space with only a change $1/m$ in the z-coordinate between them. Let that be the base of the triangle. Let Z be the midpoint of the base. Note that

$$\begin{aligned}|P_{i,j}X_{i,j}|^2 &= (\cos(u_i) - \cos(u_i + \pi/n))^2 \\ &\qquad + (\sin(u_i) - \sin(u_i + \pi/n))^2 + (-1/(2m))^2\end{aligned}$$

while

$$\begin{aligned}|P_{i,j+1}X_{i,j}|^2 &= (\cos(u_i) - \cos(u_i + \pi/n))^2 \\ &\qquad + (\sin(u_i) - \sin(u_i + \pi/n))^2 + (1/(2m))^2\end{aligned}$$

and thus are equal. So the triangles $P_{i,j}X_{i,j}Z$ and $P_{i,j+1}X_{i,j}Z$ are congruent. It follows that the angles at Z for both triangles is $\pi/2$ and thus the line segment from Z to $X_{i,j}$ is an altitude of the triangle $P_{i,j}P_{i,j+1}X_{i,j}$.

Note that

$$Z = \langle \cos(u_i), \sin(u_i), v_j + 1/(2m)\rangle$$

while

$$X_{i,j} = \langle \cos(u_i + \pi/n), \sin(u_i + \pi/n), v_j + 1/(2m)\rangle$$

Thus Z and $X_{i,j}$ lie on a horizontal circle of radius 1 at height $z = v_j + 1/(2m)$ with an angle of π/n between their radius vectors. By Lemma 1 below,

$$|X_{i,j}Z| = 2\sin(\pi/(2n)).$$

Hence the area of a "left side" triangle is

$$(1/2)(1/m)(2\sin(\pi/(2n)) = \frac{\sin(\pi/(2n))}{m}$$

The total area of all "left side" or "right side" triangles. Since we have $2mn$ of these congruent triangles, the total area of them is

$$(2mn)\cdot\frac{\sin(\pi/(2n))}{m} = 2n\sin(\pi/(2n)) \to \pi \quad \text{as } n \to \infty.$$

(See Lemma 3 below for the value of this limit.) By Lemma 2, this total area approximates half of the cylinder's surface area as an underestimate:

$$2n\sin(\pi/(2n)) \leq 2n(\pi/(2n)) = \pi.$$

Finding the area of any "top" or "bottom" triangle. The reader is asked to understand that any "top" triangle is congruent to any "bottom" triangle. Again, we need compute the area of just one of these, say an arbitrary "bottom triangle" $P_{i,j}P_{i+1,j}X_{i,j}$.

We shall view the line segment from $P_{i,j}$ to $P_{i+1,j}$ as the base of the triangle. These points are on a horizontal circle of radius 1 with an angle of $2\pi/n$ between their radius vectors. By Lemma 2 below, the length of the base is

$$|P_{i,j}P_{i+1,j}| = 2\sin(\pi/n)$$

Let Y be the midpoint of the segment from $P_{i,j}$ to $P_{i+1,j}$. Let $O = \langle 0, 0, v_j\rangle$ be the center for a horizontal circle of radius 1 that includes $P_{i,j}$ and $P_{i+1,j}$. The triangles $P_{i,j}OY$ and $P_{i+1,j}0Y$ are congruent. Their angles at O are equal, add to $2\pi/n$ and hence are π/n. Their angles at Y are $\pi/2$. Hence

$$|OY| = |OY|/1 = |OY|/|OP_{i,j}| = \cos(\pi/n)$$

Let the line segment from O to Y be extended past Y until its length is 1; let W be the end point. Then W is also on the horizontal circle that includes $P_{i,j}$ and $P_{i+1,j}$ (since $P_{i,j}$ and $P_{i+1,j}$ have third coordinate equal to v_j, as does O, the same applies to Y and W). Note that

$$|YW| = 1 - \cos(\pi/n) = 2\sin^2(\pi/(2n))$$

The radius vector from O to W makes an angle of π/n with the radius vectors to $P_{i,j}$ and to $P_{i+1,j}$. Thus $W = \langle\cos(u_i + \pi/n), \sin(u_i + \pi/n), v_j\rangle$. Since $X_{i,j}$ has the same first two coordinates as W and 3rd coordinate $v_j + 1/(2m)$, the points W and $X_{i,j}$ lie on a vertical line with distance $1/(2m)$ between them. The line segment from Y to W is horizontal; by the Pythagorean theorem,

$$|YX_{i,j}|^2 = |YW|^2 + |WX_{i,j}|^2 = 4\sin^4(\pi/(2n)) + 1/(4m^2).$$

Thus, if we can prove that the line segment from Y to $X_{i,j}$ is an altitude of the triangle $P_{i,j}P_{i+1,j}X_{i,j}$, the area of this triangle is

$$(1/2) \cdot (2\sin(\pi/n)) \cdot \sqrt{4\sin^4(\pi/(2n)) + 1/(4m^2)}$$

which simplifies to

$$\sin(\pi/n)\sqrt{4\sin^4(\pi/(2n)) + 1/(4m^2)}$$

One way to establish the altitude is to prove that $|P_{i,j}X_{i,j}| = |P_{i+1,j}X_{i,j}|$. Then the triangles $P_{i,j}X_{i,j}Y$ and $P_{i+1,j}X_{i,j}Y$ would be congruent. This would make their angles at Y each $\pi/2$ and establish the line segment from $X_{i,j}$ to Y as an altitude of the triangle $P_{i,j}P_{i+1,j}X_{i,j}$. Note that the line segment from $P_{i,j}$ to W is horizontal, as is the line segment from $P_{i+1,j}$ to W, while the line segment from W to $X_{i,j}$ is vertical. Moreover, the radius vectors from O to $P_{i,j}$ make the angle π/n with the radius vector from O to W, as do the two radius vectors from O to $P_{i+1,j}$ and to W. By Lemma 2,

$$|P_{i,j}W| = 2\sin(\pi/(2n)) = |P_{i+1,j}W|.$$

It follows that

$$\begin{aligned} |P_{i,j}X_{i,j}| &= \sqrt{|P_{i,j}W|^2 + |WX_{i,j}|^2} \\ &= \sqrt{|P_{i+1,j}W|^2 + |WX_{i,j}|^2} \\ &= |P_{i+1,j}X_{i,j}| \end{aligned}$$

The total area of all "bottom" or "top" triangles. There are $2mn$ of these congruent triangles, so their total area is

$$(2mn) \cdot \sin(\pi/n)\sqrt{4\sin^4(\pi/(2n)) + 1/(4m^2)} = n\sin(\pi/n)\sqrt{1 + 16m^2\sin^4(\pi/(2n))}$$

Suppose that we choose $n_k \to \infty$ and $m_k \to \infty$ so that $m_k/n_k^2 \to \lambda$. By Lemma 3,

$$n_k \sin(\pi/n_k) \to \pi$$

Also

$$m_k^2 \sin^4(\pi/(2n_k)) = \frac{m_k^2}{n_k^4}\left(n_k \sin(\pi/(2n_k))\right)^4 \to \lambda^2(\pi/2)^4.$$

Hence the total area of all "bottom" or "top" triangles can be arranged to converge, as $n_j \to \infty$, to

$$\pi\sqrt{1 + \lambda^2\pi^4}.$$

By choosing $\lambda = 0$ we get the true answer of π; by choosing $\lambda > 0$ we can get any limit bigger than π, including $+\infty$.

Supporting Lemmas

Lemma 1. *Let P and Q be two points of a circle of radius r and center C. Suppose the angle between the vectors $\overrightarrow{CP}$ and $\overrightarrow{CQ}$ is θ in radians (here $0 \leq \theta \leq \pi$). Then the Euclidean distance from P to Q is $2r\sin(\theta/2)$.*

Proof. Let M be the midpoint of the line segment between P and Q. The triangles CMQ and CMP are congruent and hence the angles at C for each of these triangles are equal. These angles add to θ; thus each is equal to $\theta/2$. The angle at M for both triangles is $\pi/2$ and thus

$$\sin(\theta/2) = \frac{|MQ|}{|CQ|} = \frac{|MQ|}{r} \quad \text{and hence} \quad |MQ| = r\sin(\theta/2).$$

Here $|AB|$ denotes the Euclidean distance from A to B. Since M is the midpoint of the line segment from P to Q, we have $|PQ| = 2|MQ| = 2r\sin(\theta/2)$. □

Lemma 2. *For $\theta \geq 0$, we have $\sin(\theta) \leq \theta$.*

Proof. Let $h(\theta) = \theta - \sin(\theta)$. Then $h'(\theta) = 1 - \cos(\theta) \geq 0$ for all θ. Thus h is increasing on $\mathbb{R}$ in the sense that $a < b$ implies $h(a) \leq h(b)$. Note that $h(0) = 0$. Thus $0 \leq h(\theta)$ for $\theta > 0$. □

Lemma 3. *For any C,*

$$\lim_{n\to\infty} n\sin(C/n) = C.$$

Proof. One can use the fact that $\lim_{t\to 0}\sin(t)/t = 1$. Then

$$\lim_{n\to\infty} n\sin(C/n) = \lim_{n\to\infty} C\frac{\sin(C/n)}{C/n} = C \cdot 1 = C.$$

□