We define a roadmap to be a nonempty finite collection of possibly curved line segments in a plane, each with exactly two endpoints, such that if any pair of these line segments happen to touch, then they only touch at their endpoints. For example, here is a picture of a roadmap:

![Roadmap Image](image)

(the endpoints are indicated by heavy dots), but the following are not:

![Invalid Roadmap Images](image)

Because of the resemblance of a roadmap to a streetmap of a city (without overpasses!), we will call the line segments streets, the endpoints corners, and the regions into which the streets divide the plane blocks. We will usually let $S$, $C$ and $B$ denote the number of streets, corners, and blocks, respectively, in a given roadmap. For example, in the example of a roadmap given above, $S = 6$, $C = 5$, and $B = 3$ (the region "outside" the roadmap is one of the blocks).
The number of streets with a given corner as endpoint is called the degree of that corner. Thus in the above example the corners have degrees 1, 3, 3, 4 and 1 (proceeding from left to right). A roadmap is connected if given any two corners, you can trace a path from one corner to the other without ever leaving the streets of the roadmap. The earlier example of a roadmap is connected, but the next one is not:

Given a roadmap, there are many questions you can ask about it:

1) Can you trace a path on the roadmap that goes over every street exactly once? ("Highway inspector problem")

2) Can you trace a path on the roadmap which touches every corner exactly once? ("Travelling salesman's problem")

3) What is the smallest number of colors required if you want to color each block a solid color in such a way that any two blocks which both have a given street on their boundaries always have different colors?

4) We can also ask questions about the existence of roadmaps satisfying various conditions, e.g., is there a roadmap with exactly five corners such that every pair of corners is connected by a street?

A general answer to the first question is easy to formulate.

Theorem. The highway inspector problem has an affirmative answer for a given roadmap if and only if there are at most two corners with odd degree and the roadmap is connected.
There is no corresponding result for the travelling salesman's problem; it is much more difficult. The third problem is much better, but still imperfectly, understood. Here is the major result.

**Theorem.** You never need to use more than four colors!

The proof of the above theorem settled a question that had intrigued and frustrated mathematicians for over a century. The proof is very difficult. It is not so hard to prove that five colors will suffice, if you use the following important fact.

**Theorem (Euler's formula).** For any connected roadmap, \( S + 2 = B + C \).

Euler's formula can also be used to prove the next result (cf., question #4).

**Theorem.** There is no roadmap with five corners and every pair of corners connected by a street. There is also no roadmap with six corners, say \( A, B, C, D, E, F \), such that each of the three corners \( A, B \) and \( C \) is connected with each of the three corners \( D, E \) and \( F \) by a street.

The above theorem implies that there is no roadmap with more than 5 corners such that every pair of corners is connected by a street. After all, if there were, then by deleting some streets and corners we would get a roadmap of the type whose existence is denied in the above theorem. This observation can be generalized and refined. Indeed a striking result of Kuratowski says that the above theorem describes, in a technical sense, the only obstructions to constructing a roadmap with prescribed collections of corners and of pairs of corners to be joined by a street.
We now turn to an application of Euler's formula to polyhedra.

A **polygon** is a solid figure in the plane whose boundary consists of a finite number of straight line segments. If a polygon has \( n \) "sides" (i.e., the boundary consists of \( n \) line segments), then we call it an "\( n \)-gon". 3–gons, 4–gons, 5–gons and 6–gons are more commonly known as triangles, quadrilaterals, pentagons, hexagons, and so on. A **polyhedron** is a solid figure in space whose surface consists of a finite number of polygons. Examples include:

- tetrahedrons
- cubes
- octahedrons
- and pyramids of various sorts (including tetradedra):
A polyhedron is called convex if it has no "indentations"; more precisely, if any line segment joining two points of the polyhedron lies entirely in the polyhedron. For any given polyhedron we often let $E$ denote the number of edges, $V$ denote the number of vertices (i.e., corners) and $F$ denote the number of faces. For example, in the tetrahedron $E = 6$, $F = 4$ and $V = 4$. From Euler's formula for roadmaps we can prove the following.

**Theorem (Euler's formula).** For any convex polyhedron, $E + 2 = F + V$.

A polyhedron is called regular if it is convex and if

(i) every face has the same number of edges (we let "a" denote this number), and

(ii) the number of edges ending in a given vertex is the same for every vertex (we let "b" denote this number).

For example, a tetrahedron is a regular polyhedron with $a = 3$ and $b = 3$. A cube is regular with $a = 4$ and $b = 3$. On the other hand neither of the following is regular:
Theorem. Every regular polyhedron is one of the following five types:

1) a tetrahedron;
2) a "deformed cube" (six faces, all of which are quadrilaterals with three faces meeting at each vertex);
3) an octahedron (eight faces, all triangles, with four meeting at each vertex);
4) a dodecahedron (twelve faces, all pentagons, with three meeting at each vertex);
5) an icosahedron (twenty faces, all triangles, with five meeting at each vertex).

Moreover there exist examples of each of these five types of polyhedra.

(The examples whose existence is asserted above can be constructed so that all the faces are congruent regular polygons.)