The minimal number with a given number of divisors

Ron Brown

Department of Mathematics, 2565 McCarthy Mall, University of Hawaii, Honolulu, HI 96822, USA

Phone (808) 956-4665

Fax (808) 956-9139

Email address: ron@math.hawaii.edu

Abstract

A number n is said to be *ordinary* if the smallest number with exactly n divisors is $p_1^{q_1-1} \cdots p_a^{q_a-1}$ where $q_1 \cdots q_a$ is the prime factorization of n and $q_1 \geq \ldots \geq q_a$ (and where p_k denotes the k-th prime). We show here that all square–free numbers are ordinary and that the set of ordinary numbers has natural density one.

Key words: minimal number, ordinary number, divisor, square-free, natural density

MSC : Primary 11A25, Secondary 11A41

1 Two Theorems

In 1968 M. E. Grost [2] introduced the function $A(n)$ which assigns to each number n the smallest number with exactly n divisors. (The numbers and divisors we consider here are always assumed to be positive integers.) His calculation of $A(n)$ for all numbers n which are products of 6 or fewer primes suggests that most numbers are "ordinary" in the sense of the following definition.

Definition. A number n with prime factorization $q_1 q_2 \cdots q_a$ where $q_1 \geq \ldots \geq q_a$ is called *ordinary* [2] if $A(n) = p_1^{q_1-1} \cdots p_a^{q_a-1}$ where we denote the k-th prime by p_k. A number which is not ordinary is called *extraordinary.* We regard the number 1 as being ordinary.

For example 5 is ordinary (since $A(5) = 2^{5-1} = 16$) and 8 is extraordinary (since $A(8) = 24$, not 30). There are infinitely many extraordinary numbers; indeed Grost shows that all numbers of the form $16p$ where p is a prime greater than 3 are extraordinary and one can show that a prime power p^k is extraordinary if and only if $2^p \leq p_k$. However the next two theorems indicate that Grost's terminology is well-chosen. The first shows that a large natural class of numbers is ordinary; the second gives a sense in which almost all numbers are ordinary. We will let $\mathcal{O}$ denote the set of ordinary numbers; also, we let $|A|$ denote the number of elements in a finite set A.

Theorem 1 *All square–free positive integers are ordinary.*

Theorem 2 $\mathcal{O}$ *has natural density* 1*; that is,*

$$\lim_{N\to\infty} \frac{|\mathcal{O} \cap \{1,\ldots,N\}|}{N} = 1.$$

Theorem 2 is an immediate consequence of Theorem 7 below, which gives an upper bound for $|\{1,\ldots,N\} \setminus \mathcal{O}|$. The proofs of these theorems are elementary and will assume nothing beyond the material in [5].

2 Distribution of Primes

We will assume throughout the paper that α and β denote positive constants such that for all numbers $N \geq 2$,

$$\alpha N \log N \leq p_N \leq \beta N \log N\,. \tag{1}$$

The next lemma provides the information about the distribution of primes needed in the proof of the first theorem. In the proof of this lemma we will assume that α and β are chosen in formula (1) so that $\beta/\alpha = 256$. (The proofs of [5, Theorem 8.1, p. 217 and Theorem 8.2, p. 220]—plus an easy computation—show that we may take $\alpha = 1/(32 \log 2)$ and $\beta = 8/\log 2$.) The arguments below could be simplified, but only a little, by using values of α and β closer to 1. A survey of such values as of 1996 appears in [7, p. 249].

Lemma 3 *If a and b are positive integers, then* $p_{a+b-1} < p_b^{\sqrt{4+p_a}}$.

Proof. We will often use the inequality $p_{c+d-1} < 2^{d-1} p_c$, which follows from Bertrand's Postulate [5, Theorem 8.6, p. 224]. As an application note that the lemma is true whenever $a \le 5$ and $b \ge 2$ since in these cases

$$2^{a-1} < 3^{\sqrt{4+p_a}-1} \le p_b^{\sqrt{4+p_a}-1}.$$

The lemma can also be directly verified in the ten cases with $a \le 5$ and $b \le 2$. Hence we may suppose without loss of generality that $a \ge 6$.

Now suppose $b \le 10$. The function $f(y) = p_b^y - 2^{b-1}(y^2 - 4)$ is positive on the interval $[4, \infty)$ since $f(4) > 0$, $f'(4) > 0$ and $f''(4) > 0$ and for all $y \ge 4$, $f'''(y) > 0$. If we set $y = \sqrt{p_a + 4}$ (so that $y \ge 4$ since $a \ge 6$), then we obtain

$$p_{a+b-1} < 2^{b-1} p_a = 2^{b-1}(y^2 - 4) < p_b^y = p_b^{\sqrt{p_a+4}}.$$

Therefore we may henceforth assume that $b \ge 11$, so $p_b \ge 31$.

Let $g(y) = 31^y - 3\frac{\beta}{\alpha}(y^2 - 4)$. Then g is positive on $[2, \infty)$ (argue as above). Note that since $a > 4$, we have $2 \log 2a < 3 \log a$. If $a \ge b$ we have (setting $y = \sqrt{4 + \alpha a \log a}$, so $y > 2$)

$$p_{a+b-1} < p_{2a} \le \beta(2a \log 2a) < \frac{3\beta}{\alpha}(\alpha a \log a)$$

$$= \frac{3\beta}{\alpha}(y^2 - 4) < 31^y \le p_b^{\sqrt{4+\alpha a \log a}} \le p_b^{\sqrt{4+p_a}}.$$

On the other hand, if $a < b$ then we also have

$$p_{a+b-1} < p_{2b} < 2\beta b \log 2b < 4\beta b \log b < 31^3 \alpha b \log b \le p_b^4 \le p_b^{\sqrt{4+p_a}}$$

(recall that $\beta/\alpha = 256$). This completes the proof of the lemma. □

3 Proof of Theorem 1: square free numbers are ordinary

We next introduce some more notation and terminology which will be used in the rest of the paper and make some basic observations. We let n denote an integer larger than 1, and write $n = q_1 \cdots q_a$ where $q := (q_1, \ldots, q_a)$ is a nonincreasing sequence of primes. Let $\mathcal{S}$ be the set of all finite nonincreasing sequences of integers larger than one whose product is n, so $q \in \mathcal{S}$. Given $t = (t_1, \ldots, t_d) \in \mathcal{S}$ we set $A(t) = p_1^{t_1-1} \cdots p_d^{t_d-1}$, so n is ordinary if and only if $A(n) = A(q)$. We will say that $t^* \in \mathcal{S}$ is a *compression* of t if it is obtained from t by replacing two coordinates t_s and t_r by their product $t_s t_r$. Thus for example if $n = 120$, then $(5, 4, 3, 2)$ is a compression of $(5, 3, 2, 2, 2)$. We will write $A(n) = p_1^{u_1-1} \cdots p_b^{u_b-1}$ where $u_b > 1$. It is an easy exercise [2, Lemma 3, p. 725] to verify that $u := (u_1, \ldots, u_b) \in \mathcal{S}$.

Proof of Theorem 1. Suppose that $t^* \in \mathcal{S}$ is a compression of $t = (t_1, \ldots, t_d) \in \mathcal{S}$, say obtained by replacing the coordinates t_r and t_s (where $s < r$) by their product. It suffices to show that $A(t^*) > A(t)$. After all, if $q = u$ then we are done, and otherwise we can get from q to u by a finite sequence of compressions, each of which will have greater value under the map A than the previous one, contradicting that $A(n) = A(u)$.

Let m be least with $t_m < t_s t_r$ (so $m \le s$ and if $m \ne 1$ then $t_{m-1} > t_r t_s$). Then t^* is the $(d-1)$–tuple $(t_1, \ldots, t_{m-1}, t_r t_s, t_m, \ldots, t_{s-1}, t_{s+1}, \ldots, t_{r-1}, t_{r+1}, \ldots, t_d)$ where it is understood that the subsequence $t_1, \ldots, t_{m-1}$ is empty if $m = 1$; that $t_m, \ldots, t_{s-1}$ is empty if $m = s$; that $t_{s+1}, \ldots, t_{r-1}$ is empty if $r = s + 1$; and that $t_{r+1}, \ldots, t_d$ is empty if $r = d$. For each i from 0 to $d - m$ we can pick a prime e_i dividing t_{m+i}; since the e_i are clearly distinct, we have

$$t_m \ge \max\{e_0, \ldots, e_{d-m}\} \ge \max\{p_1, \ldots, p_{d-m+1}\} = p_{d-m+1}\,.$$

Therefore

$$t_s^2 = t_s t_r + t_s(t_s - t_r) \ge (t_m + 1) + 3 \ge 4 + p_{d-m+1}$$

so the preceding lemma implies that $p_d < p_m^{t_s}$.

The above inequality $p_d < p_m^{t_s}$ implies that

$$p_r^{t_r - t_{r+1}} p_{r+1}^{t_{r+1} - t_{r+2}} \cdots p_{d-1}^{t_{d-1} - t_d} p_d^{t_d - 1} \le p_d^{t_r - 1}$$

$$< p_m^{t_s t_r - t_s} \le p_m^{t_r t_s - t_m} p_{m+1}^{t_m - t_{m+1}} \cdots p_s^{t_{s-1} - t_s}\,.$$

Therefore $p_1^{t_1 - 1} \cdots p_d^{t_d - 1}$ is less than

$$p_1^{t_1-1}\cdots p_{m-1}^{t_{m-1}-1}p_m^{t_r t_s-1}p_{m+1}^{t_m-1}\cdots p_s^{t_{s-1}-1}p_{s+1}^{t_{s+1}-1}\cdots p_{r-1}^{t_{r-1}-1}p_r^{t_{r+1}-1}\cdots p_{d-1}^{t_d-1},$$

i.e., $A(t) < A(t^*)$, which was to be proved. □

Remark 4 The proof of Theorem 1 can be adapted to show that all numbers of the form $4n$ where n is square-free and odd are also ordinary.

4 Proof of Theorem 2: almost all numbers are ordinary

We continue to let the unmodified noun "number" mean positive integer, but we will also refer to "real numbers" in this section which are not necessarily integers. For any real number x we let $\lceil x \rceil = -[-x]$ denote the smallest number N with $N \geq x$.

The proof of Theorem 2 depends in part on the idea that a number which is not square-free can still be ordinary if it has many distinct prime factors relative to its total number of prime factors. The next lemma shows one way in which this can happen.

Lemma 5 *For each real number x let $F(x)$ denote the set of numbers not divisible by any $\lceil x \rceil$–th power of a prime and let $P(x)$ denote the set of products of at least $\lceil x \rceil$ primes. Then for any real number $K \geq 1$ there exists a real number D with $F(K) \cap P(D) \subseteq \mathcal{O}$.*

$P(2)$ is of course the set of composite numbers. $F(2)$ is the set of square–free numbers, so Theorem 1 says that if $K = 2$, then we can take $D = 1$ in the above lemma.

Remark 6 The proof of Lemma 5 will show that for any real number $K \geq 1$, it suffices to pick $D > 4K^2$ large enough that for all $z \geq \sqrt{D}/2$ we have

$$2^{\sqrt{\alpha(z/K)\,\log(z/K)}} \;>\; 2\beta z \log(2z) \tag{2}$$

and

$$(\alpha z\,\log z)^2 \;>\; 2\beta z \log(2z)\,. \tag{3}$$

Proof. Pick a real number $D > 4K^2$ satisfying the conditions of Remark 6.

Suppose $n \in F(K) \cap P(D)$. As in Section 3 we assume that $A(n) = A(u)$; note that $a \geq D > 4$. For each $j \leq b$ we can write $u_j = pq$ where p is prime. Then by the minimality of $A(n) = A(u)$ we must have

$$p_j^{pq-1} < p_j^{q-1} p_{b+1}^{p-1}\,,$$

so $p_j^q < p_{b+1}$. Hence by Bertrand's postulate

$$2^{q-1} p_j \leq p_j^q < p_{b+1} < 2^{b-j+1} p_j\,,$$

so $q \leq b - j + 1$. Therefore u_j has at most $b - j + 1$ prime factors. Thus $n = u_1 \cdots u_b$ has at most $\frac{b^2+b}{2} \leq b^2$ prime factors, so $b \geq \sqrt{a}\,$.

Now consider any $t = (t_1, \ldots, t_d) \in \mathcal{S}$ with $d \geq \sqrt{a}$ (so $d/2 \geq \sqrt{D}/2$). Let t^* be a compression of t obtained by replacing t_r and t_s (where $r > s$) by $t_r t_s$. There exists a least m with $t_m \leq t_r t_s$. By definition $t_m \geq t_{m+1} \geq \cdots \geq t_d$; consequently, t_m is at least as large as the largest prime dividing $T := t_m \cdots t_d$. Note that T is a product of at least $d-m+1$ primes and since $n \in F(K)$ each one of these primes is repeated fewer than $\lceil K \rceil$ times. Thus T has at least $g := 1 + \left[\frac{d-m}{K}\right]$ distinct prime factors. Therefore if $m < d/2$, then $d - m > d/2 \geq \sqrt{D}/2 > K$, and

$$t_s^2 \geq t_s t_r \geq t_m \geq p_g \geq \alpha \frac{d-m}{K} \log \frac{d-m}{K}\,.$$

Hence by the choice of D we have

$$p_m^{t_s} \geq 2^{\sqrt{\alpha(d/2K)\log(d/2K)}} > 2\beta(d/2)\log d \geq p_d\,.$$

The inequality $p_m^{t_s} > p_d$ also holds if $m \geq d/2$ since then by the choice of D

$$p_m^{t_s} \geq \left(\alpha \frac{d}{2} \log \frac{d}{2}\right)^2 \geq \beta d \log d \geq p_d\,.$$

From the inequality $p_m^{t_s} > p_d$ we can conclude that $A(t^*) > A(t)$ (argue exactly as in the last paragraph of the proof of Theorem 1). It follows as in the proof of Theorem 1 that n is ordinary, since otherwise we can obtain u from q by a proper finite sequence of compressions of elements of $\mathcal{S}$ whose number of coordinates is greater than or equal to $\sqrt{a}$, each compression having an increased value under A, contradicting that $A(q) = A(u)$. $\square$

Theorem 2 follows immediately from the next result.

Theorem 7 *Suppose that* $0 < \delta < \frac{1}{2}$. *Then*

$$|\{1, 2, \cdots, N\} \setminus \mathcal{O}| = o(N/2^{(\log\log N)^\delta}).$$

In the proof below the symbol p will always be understood to denote a prime number.

Proof of Theorem 7. We will show for sufficiently large numbers N that $|\{1, 2, \cdots, N\} \setminus \mathcal{O}|$ is bounded by a sum of two functions, each of which is $o(N/2^{(\log\log N)^\delta})$. First pick $\epsilon \in (0, \frac{1}{2} - \delta)$. Consider any N large enough that the inequalities (4) and (7) below are valid. Then set $K = (\log\log N)^{\frac{1}{2}-\epsilon}$ and $D = (\log\log N)^{1-\epsilon}$. We now verify that D and K satisfy the conditions of Remark 6.

There exists a number $y_0 > 2$ such that for all real numbers $z > y_0$,

$$2^{\sqrt{\alpha z\,\log z}} > (\alpha z\,\log z)^2 \;>\; 2\beta z^2 \log(2z^2)\,.$$

We require N to be large enough that

$$(\log\log N)^{\epsilon/2} > 2y_0\,. \tag{4}$$

Thus $K > 1$ and $D > 4K^2$. Suppose that $x \geq \sqrt{D}/2 = (\log\log N)^{\frac{1}{2}-\frac{\epsilon}{2}}/2$. Since $1/2 > \epsilon$ we have $x \geq (\log\log N)^{\frac{1}{2}-\frac{\epsilon}{2}}/2 > (\log\log N)^{\frac{\epsilon}{2}}/2 > y_0$, so that the inequality (3) of Remark 6 is valid for $z = x$. Now let $y = x/K$; then

$$y > (\sqrt{D}/2)/K = (\log\log N)^{\epsilon/2}/2 > y_0\,. \tag{5}$$

Hence if $y > K$, then by the choice of y_0 we have

$$\alpha(\log 2)^2 y \log y > (\log(2\beta y K \log(2yK)))^2. \tag{6}$$

If we choose N large enough that

$$\alpha(\log 2)^2((\log\log N)^{\epsilon/2}/2)\log((\log\log N)^{\epsilon/2}/2)$$

$$> (\,\log(2\beta(\log\log N)^{1-2\epsilon}\log(2(\log\log N)^{1-2\epsilon})))^2\,, \tag{7}$$

then using the inequality (5) we see that (6) holds also in the case that $y \leq K$. Thus in all cases the inequality (2) of Remark 6 holds with $z = yK = x$. Thus K and D satisfy the conditions of Remark 6 and hence

$$|\{1,\ldots,N\} \setminus \mathcal{O}| \leq |\{1,\ldots,N\} \setminus P(D)| + |\{1,\ldots,N\} \setminus F(K)|\,. \tag{8}$$

We next show that both summands on the right hand side of the inequality (8) are $o(N/2^{(\log\log N)^\delta})$. For the second summand we have

$$|\{1,\ldots,N\} \setminus F(K)| \leq \sum_{p\geq 2}\left[\frac{N}{p^{\lceil K\rceil}}\right] \leq N\sum_{p\geq 2}\frac{1}{p^K}$$

$$\leq N\left(\frac{1}{2^K} + \int_2^\infty \frac{dx}{x^K}\right) = \frac{N}{2^K}\,\frac{K+1}{K-1} = o\left(\frac{N}{2^{(\log\log N)^\delta}}\right)$$

since $K = (\log\log N)^{\frac{1}{2}-\epsilon}$ and $1/2 > \epsilon + \delta$.

Pick real numbers C and E such that for all $x \geq 3$ we have $1+\pi(x) \leq Cx/\log x$ and $\sum_{p\leq x} \frac{1}{p} \leq E \log\log x$ [5, Theorem 8.1, p. 217, and Theorem 8.4, p. 222]. We now show that for all numbers $L \geq 2$ and $M \geq e^e$ we have

$$|\{1,\dots,M\} \setminus P(L)| \leq C\frac{M}{\log M}(8E\log\log M)^{L-2}\,. \tag{9}$$

The inequality (9) is obvious if $L = 2$; suppose that it is valid for some $L \geq 2$. We can write

$$|\{1,\dots,M\} \setminus P(L+1)| \leq 1 + \pi(M) + \sum_p |\{1,\dots,[M/p]\} \setminus P(L)|$$

where we sum over all primes $p \leq \sqrt{M}$. Note that $8E\log\log M \geq 2$ and that if $p \leq \sqrt{M}$, then $\log[M/p] > \frac{1}{4}\log M$. Hence by the induction hypothesis

$$\begin{aligned}|\{1,\dots,M\} \setminus P(L+1)| &\leq C\frac{M}{\log M} + C\frac{4M}{\log M}\,(8E\log\log M)^{L-2}\sum_{p\leq M}\frac{1}{p}\\ &\leq C\frac{M}{\log M}(8E\log\log M)^{L-1}\,,\end{aligned}$$

proving the inequality (9).

Since $P(D) = P(\lceil D\rceil)$, $\lceil D\rceil - 2 < D$ and $N > e^e$, we may deduce from the inequality (9) that

$$|\{1,\dots,N\} \setminus P(D)| \leq C\frac{N}{\log N}(8E\log\log N)^D$$

which is $o(N/2^{(\log\log N)^\delta})$ because

$$\log\left(C\frac{N}{\log N}(8E\log\log N)^D \Big/ \frac{N}{2^{(\log\log N)^\delta}}\right)$$

$$= \log C - \log\log N + (\log\log N)^{1-\epsilon} \log(8E \log\log N) + (\log 2)(\log\log N)^{\delta}$$

goes to $-\infty$ as $N \to \infty$. $\square$

Remark 8 The inequality (9) can be improved when L is sufficiently small relative to M. There exist absolute constants A and B such that if $2 \leq L \leq (\log\log M)/2$, then

$$|\{1, \ldots, M\} \setminus P(L)| \leq A \frac{M}{\log M} \frac{(B + \log\log M)^{L-2}}{(L-2)!} . \tag{10}$$

After all, any number less than or equal to M with fewer than L prime factors has at most $L - 1$ distinct prime factors; the bounds of [3, Lemma B, p. 85] therefore can be applied to give the inequality (10). (The restriction on L guarantees that

$$\frac{(L-2)!}{(B + \log\log M)^{L-2}} \sum_{k=0}^{L-2} \frac{(B + \log\log M)^k}{k!} \leq \sum_{k=0}^{L-2} \frac{1}{2^k} < 2 .)$$

Remark 9 Grost calls numbers in the range of A *minimal numbers*. Ramanujan's highly composite numbers [6] (*i.e.,* numbers with more divisors than any smaller number) are all minimal numbers but the converse is false. For example, $A(5) = 16$ is minimal, but it is not highly composite since 12 has more divisors. It would be interesting to study the asymptotic behavior of minimal numbers and to compare this behavior to that of the highly composite numbers [1], [4], [6]. In particular, it would be interesting to have a sense of what proportion of minimal numbers are highly composite.

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