Nonstandard Analysis basics for seminar

(I) Nonstandard Analysis

Start with a mathematical universe (*superstructure*) V, containing:

- All natural numbers $0, 1, 2, \ldots$; real numbers $\sqrt{2}, \pi, e, \phi, \ldots$; etc.
- The *set* $\mathbb{N}$ of natural numbers *as an object*; the set $\mathbb{R}$ of real numbers; etc.
- Every function from $\mathbb{R}$ to $\mathbb{R}$, and the set of all such functions
- Your favorite groups, Banach spaces, measure spaces, etc
- Every other mathematical object we might want to talk about
- Closure under $\in, \mathcal{P}$, etc.
- We call the elements of this mathematical universe *standard*.

Extend to a *nonstandard* mathematical universe $^*V$:

- For every object A in V, there is a corresponding object $^*A$ in $^*V$
- EG, $^*V$ has objects $^*\mathbb{N}$, $^*\mathbb{R}$, $^*\sin(x)$, etc.
- (For simplicity, we drop the stars from simple objects like numbers: 12 instead of ^{*12} etc)
- There may (generally will) be many more objects in $^*V$ than in V
- An element of $^*V$ that is **not** in V is called *nonstandard*.

The extension should satisfy two important properties:

Transfer If S is a bounded first-order statement about objects in V, then S is true in V if and only if it true in $^*V$

For example, let $(G, \cdot, e)$ be a multiplicative group; the following are true in V:

$(\forall x \in G)(\exists y \in G)[(x \cdot y = e) \wedge (y \cdot x = e)]$

$(\forall x \in G)[(x \cdot e = x) \wedge (x \cdot e = x)]$

$(\forall x \in G)(\forall y \in G)(\forall z \in G)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$

By transfer it follows:

$(\forall x \in^* G)(\exists y \in^* G)[(x {}^*\!\cdot y =^* e) \wedge (y {}^*\!\cdot x =^* e)]$

$(\forall x \in^* G)[(x {}^*\!\cdot {}^*e = x) \wedge (x {}^*\!\cdot {}^*e) = x)]$

$(\forall x \in^* G)(\forall y \in^* G)(\forall z \in^* G)[(x {}^*\!\cdot y) {}^*\!\cdot z = x {}^*\!\cdot (y {}^*\!\cdot z)]$

In other words, $^*G$ is also not only a *group, but also an actual group.

As another example, since 12 is an element of $\mathbb{N}$, ^{*12} is an element of $^*\mathbb{N}$.

Since we can think of the basic elements (like ^{*12}) of $^*V$ as just being the same as their counterparts (like 12) in V, $^*\mathbb{N}$ is a superset of $\mathbb{N}$.

Similarly, for *any* standard set A which is an object of V, the set $^*A$ in $^*V$ extends the set A.

Saturation:

A set $a \subseteq^* V$ is *internal* if $\exists b \in V \;\; a \in^* b$ (otherwise it is *external*)

For example, if $A \in V$ then $\mathcal{P}(A) \in V$, so $^*A \in^* \mathcal{P}(A)$ holds, and $^*A$ is internal.

Equivalently, a set a is internal if it can be defined from other internal sets by a bounded first-order formula.

Now, κ-saturation is the property:

If $\mathcal{A}$ is a family of sets with the finite intersection property, and $|\mathcal{A}| < \kappa$, then $\bigcap \mathcal{A} \neq \emptyset$.

Equivalently, any set of statements of cardinality $< \kappa$ about an object X which is finitely satisfiable in $^*V$, can all be simultaneously satisfied by a single object in $^*V$

We will always assume that the model is κ—saturated for κ bigger than the cardinality of every standard set (though much less saturation usually suffices).

Saturation roughly means: Anything that can happen in $^*V$, *does* happen.

Example: Consider the statements:

$$x \text{ is a real number}$$

$$x > 0$$

$$x < 1$$

$$x < 1/2$$

$$x < 1/3$$

$$x < 1/4$$

$$\vdots$$

Any finite set of these statements refers to a smallest fraction $1/N$; but then, $x = \frac{1}{N+1}$ satisfies this finite set of statements.

It follows that there is a an element of $^*\mathbb{R}$, call it ϵ, such that

$$\epsilon > 0$$

and, for every (standard) natural number N,

$$\epsilon < 1/N$$

We have proved that $^*\mathbb{R}$ contains nonzero infinitesimals, where

Definition: An *infinitesimal* is an element ϵ of $^*\mathbb{R}$ such that

$$|\epsilon| < 1/N$$

for every natural number N in $\mathbb{N}$

Since $^*\mathbb{R}$ (sometimes called the set of *"hyperreal numbers"*) is, like the usual set of real numbers, closed under the basic arithmetic operations, it also contains negative infinitesimals (like $-\epsilon$), infinite numbers (like $1/\epsilon$), and many other objects.

In particular, as we have seen there are elements of ${}^*\mathbb{N}$ which are bigger than every element of $\mathbb{N}$; in other words, there are *infinite integers*.

The set $\{x \in {}^*\mathbb{R} | \exists N \in \mathbb{N}\ \ |x| < N\}$ is the set of *finite* elements of ${}^*\mathbb{R}$. It is the same as the set of *nearstandard* elements of $\mathbb{R}$, namely the set of $x \in {}^*\mathbb{R}$ such that for some standard $x_0 \in \mathbb{R}, (x - x_0)$ is infinitesimal. This x_0 is unique if it exists, and we denote it by ${}^\circ x$ or $st(x)$, the *standard part* of x.

The set of all infinitesimals, and the set of all finite numbers, are both **external** subrings (but not subfields) of $\mathbb{R}$, and the standard part map $x \mapsto {}^\circ x$ is a ring homomorphism.

Many applications are based on the ubiquity of "hyperfinite sets":

Definition: A set E in ${}^*V$ is *hyperfinite* if there is a **one-to-one correspondence* between E and $\{0, 1, 2, \ldots, H\}$ for some H in ${}^*\mathbb{N}$. Equivalently, if the mathematical statement "E is finite" holds in ${}^*V$.

Examples: 1. Every finite set is hyperfinite.

2. If H is an infinite integer, $\{0, 1, 2, \cdots, H\} = \{n \in {}^*\mathbb{N} : n \leq H\}$ is a hyperfinite subset of ${}^*\mathbb{N}$

3. If H is an infinite integer, $\{0, \frac{1}{H}, \frac{2}{H}, \cdots, \frac{H-1}{H}, 1\}$is a hyperfinite subset of ${}^*[0.1]$

Theorem: If A is an infinite set in V then there is a hyperfinite set $\hat{A}$ in ${}^*V$ such that every element of A is in $\hat{A}$

Proof: Consider the statements: (i) X is finite; (ii) $a \in X$ (one such statement for every element a of A)

Given any finite number of these statements, a corresponding finite number $\{a_1, \ldots, a_n\}$ of elements of A are mentioned, so $X = \{a_1, \ldots, a_n\}$ satisfies those statements. By the saturation principle there is therefore a set X in ${}^*V$ satisfying all the statements simultaneously; let $\hat{A}$ be this X. $\dashv$

Corollary: There is a hyperfinite set containing $\mathbb{R}$.

> "Nonstandard analysis is the art of making infinite sets finite by extending them." —M. Richter

(II) Loeb Measures

- Let $(\Omega, \mathcal{A}, \mu)$ be an internal finitely additive finite *-measure. (This means that Ω is an internal set, $\mathcal{A}$ is an internal *−algebra on Ω, and $\mu : \mathcal{A} \rightarrow^* [0, \infty)$ is an internal function satisfying (i) $\mu(\emptyset) = 0$, (ii) $\mu(\Omega)$ is finite, and and (iii) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$ are disjoint.)

- Note that $\mathcal{A}$ is (externally) an algebra on Ω, and $st \circ \mu = {}^{\circ *}\mu$ is an actual finitely-additive measure on $(\Omega, \mathcal{A})$.

- Moreover, if $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ is a sequence of elements of $\mathcal{A}$ indexed by the standard natural numbers, and the intersection $\bigcap_n A_n$ is empty, then by $\aleph_1$−saturation there is a finite N such that $\bigcap_{n \leq N} A_n = \emptyset$.

- The Carathéodory extension criterion is therefore satisfied trivially, and $(\Omega, \mathcal{A}, {}^{\circ}\mu)$ extends to a countably-additive measure space $(\Omega, \mathcal{A}_L, \mu_L)$, (a *Loeb space*) where $\mathcal{A}_L$ is the smallest (external) sigma-algebra containing $\mathcal{A}$.

- **A useful fact:** If $E \in \mathcal{A}_L$, and $\epsilon > 0$ is standard, then $\exists A_i, A_o \in \mathcal{A}$ such that $A_i \subseteq E \subseteq A_o$ and $\mu(A_o) - \mu(A_i) < \epsilon$,