Nonstandard Analysis basics for seminar

(I) Nonstandard Analysis

Start with a mathematical universe \((\text{superstructure})\ V\), containing:

- All natural numbers \(0,1,2,\ldots\); real numbers \(\sqrt{2},\pi,e,\phi,\ldots\); etc.
- The set \(\mathbb{N}\) of natural numbers as an object; the set \(\mathbb{R}\) of real numbers; etc.
- Every function from \(\mathbb{R}\) to \(\mathbb{R}\), and the set of all such functions
- Your favorite groups, Banach spaces, measure spaces, etc
- Every other mathematical object we might want to talk about
- Closure under \(\epsilon, \mathcal{P}\), etc.
- We call the elements of this mathematical universe standard.

Extend to a nonstandard mathematical universe \(\ast V\):

- For every object \(A\) in \(V\), there is a corresponding object \(\ast A\) in \(\ast V\)
- EG, \(\ast V\) has objects \(\ast \mathbb{N}, \ast \mathbb{R}, \ast \sin(x)\), etc.
- (For simplicity, we drop the stars from simple objects like numbers: \(12\) instead of \(\ast 12\) etc)
- There may (generally will) be many more objects in \(\ast V\) than in \(V\)
- An element of \(\ast V\) that is not in \(V\) is called nonstandard.

The extension should satisfy two important properties:

Transfer If \(S\) is a bounded first-order statement about objects in \(V\), then \(S\) is true in \(V\) if and only if it true in \(\ast V\).

For example, let \((G, \cdot, e)\) be a multiplicative group; the following are true in \(V\):
\[(\forall x \in G)(\exists y \in G)[(x \cdot y = e) \land (y \cdot x = e)]\]
\[(\forall x \in G)[(x \cdot e = x) \land (x \cdot e = x)]\]
\[(\forall x \in G)(\forall y \in G)(\forall z \in G)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]\]

By transfer it follows:
\[(\forall x \in^* G)(\exists y \in^* G)[(x^* \cdot y =^* e) \land (y^* \cdot x =^* e)]\]
\[(\forall x \in^* G)[(x^* \cdot e = x) \land (x^* \cdot e = x)]\]
\[(\forall x \in^* G)(\forall y \in^* G)(\forall z \in^* G)[(x^* \cdot y^* \cdot z = x^* \cdot (y^* \cdot z)]\]

In other words, \(^*G\) is also not only a \(^*\)group, but also an actual group.

As another example, since 12 is an element of \(\mathbb{N}\), \(^*12\) is an element of \(^*\mathbb{N}\).

Since we can think of the basic elements (like \(^*12\)) of \(^*V\) as just being the same as their counterparts (like 12) in \(V\), \(^*\mathbb{N}\) is a superset of \(\mathbb{N}\).

Similarly, for any standard set \(A\) which is an object of \(V\), the set \(^*A\) in \(^*V\) extends the set \(A\).

**Saturation:**

A set \(a \subseteq^* V\) is internal if \(\exists b \in V \ a \in^* b\) (otherwise it is external)

For example, if \(A \in V\) then \(\mathcal{P}(A) \in V\), so \(^*A \in^* \mathcal{P}(A)\) holds, and \(^*A\) is internal.

Equivalently, a set \(a\) is internal if it can be defined from other internal sets by a bounded first-order formula.

Now, \(\kappa\)-saturation is the property:

If \(\mathcal{A}\) is a family of sets with the finite intersection property, and \(|\mathcal{A}| < \kappa\), then \(\bigcap \mathcal{A} \neq \emptyset\).

Equivalently, any set of statements of cardinality \(< \kappa\) about an object \(X\) which is finitely satisfiable in \(^*V\), can all be simultaneously satisfied by a single object in \(^*V\)
We will always assume that the model is $\kappa$–saturated for $\kappa$ bigger than the cardinality of every standard set (though much less saturation usually suffices).

Saturation roughly means: Anything that can happen in $\ast V$, does happen.

**Example:** Consider the statements:

- $x$ is a real number
  - $x > 0$
  - $x < 1$
  - $x < 1/2$
  - $x < 1/3$
  - $x < 1/4$
  - $\vdots$

Any finite set of these statements refers to a smallest fraction $1/N$; but then, $x = \frac{1}{N+1}$ satisfies this finite set of statements.

It follows that there is a an element of $\ast \mathbb{R}$, call it $\epsilon$, such that

$\epsilon > 0$

and, for every (standard) natural number $N$,

$\epsilon < 1/N$

We have proved that $\ast \mathbb{R}$ contains nonzero infinitesimals, where

**Definition:** An *infinitesimal* is an element $\epsilon$ of $\ast \mathbb{R}$ such that

$|\epsilon| < 1/N$

for every natural number $N$ in $\mathbb{N}$

Since $\ast \mathbb{R}$ (sometimes called the set of “hyperreal numbers”) is, like the usual set of real numbers, closed under the basic arithmetic operations, it also contains negative infinitesimals (like $-\epsilon$), infinite numbers (like $1/\epsilon$), and many other objects.
In particular, as we have seen there are elements of \( ^*\mathbb{N} \) which are bigger than every element of \( \mathbb{N} \); in other words, there are infinite integers.

The set \( \{ x \in ^*\mathbb{R} \mid \exists N \in \mathbb{N} \, |x| < N \} \) is the set of finite elements of \( ^*\mathbb{R} \). It is the same as the set of nearstandard elements of \( \mathbb{R} \), namely the set of \( x \in ^*\mathbb{R} \) such that for some standard \( x_0 \in \mathbb{R} \), \((x - x_0)\) is infinitesimal. This \( x_0 \) is unique if it exists, and we denote it by \( \sharp x \) or \( \text{st}(x) \), the standard part of \( x \).

The set of all infinitesimals, and the set of all finite numbers, are both external subrings (but not subfields) of \( \mathbb{R} \), and the standard part map \( x \mapsto \sharp x \) is a ring homomorphism.

Many applications are based on the ubiquity of “hyperfinite sets”:

**Definition:** A set \( E \) in \( ^*V \) is hyperfinite if there is a \(^*\) one-to-one correspondence between \( E \) and \( \{0, 1, 2, \ldots, H\} \) for some \( H \) in \( ^*\mathbb{N} \). Equivalently, if the mathematical statement “\( E \) is finite” holds in \( ^*V \).

**Examples:**
1. Every finite set is hyperfinite.
2. If \( H \) is an infinite integer, \( \{0, 1, 2, \ldots, H\} = \{n \in ^*\mathbb{N} : n \leq H\} \) is a hyperfinite subset of \( ^*\mathbb{N} \).
3. If \( H \) is an infinite integer, \( \{0, \frac{1}{H}, \frac{2}{H}, \ldots, \frac{H-1}{H}, 1\} \) is a hyperfinite subset of \( ^*[0, 1] \).

**Theorem:** If \( A \) is an infinite set in \( V \) then there is a hyperfinite set \( \hat{A} \) in \( ^*V \) such that every element of \( A \) is in \( \hat{A} \).

**Proof:** Consider the statements: (i) \( X \) is finite; (ii) \( a \in X \) (one such statement for every element \( a \) of \( A \)).

Given any finite number of these statements, a corresponding finite number \( \{a_1, \ldots, a_n\} \) of elements of \( A \) are mentioned, so \( X = \{a_1, \ldots, a_n\} \) satisfies those statements. By the saturation principle there is therefore a set \( X \) in \( ^*V \) satisfying all the statements simultaneously; let \( \hat{A} \) be this \( X \). \( \Box \)

**Corollary:** There is a hyperfinite set containing \( \mathbb{R} \).

“Nonstandard analysis is the art of making infinite sets finite by extending them.” —M. Richter
(II) Loeb Measures

• Let \((\Omega, \mathcal{A}, \mu)\) be an internal finitely additive finite \(^*\)-measure. (This means that \(\Omega\) is an internal set, \(\mathcal{A}\) is an internal \(^*\)-algebra on \(\Omega\), and \(\mu : \mathcal{A} \rightarrow \mathbb{R}\) is an internal function satisfying (i) \(\mu(\emptyset) = 0\), (ii) \(\mu(\Omega)\) is finite, and (iii) \(\mu(A \cup B) = \mu(A) + \mu(B)\) whenever \(A, B \in \mathcal{A}\) are disjoint.)

• Note that \(\mathcal{A}\) is (externally) an algebra on \(\Omega\), and \(\text{st} \circ \mu = \circ^* \mu\) is an actual finitely-additive measure on \((\Omega, \mathcal{A})\).

• Moreover, if \(A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots\) is a sequence of elements of \(\mathcal{A}\) indexed by the standard natural numbers, and the intersection \(\bigcap_n A_n\) is empty, then by \(\aleph_1\)-saturation there is a finite \(N\) such that \(\bigcap_{n \leq N} A_n = \emptyset\).

• The Carathéodory extension criterion is therefore satisfied trivially, and \((\Omega, \mathcal{A}, \mu)\) extends to a countably-additive measure space \((\Omega, \mathcal{A}_L, \mu_L)\), (a Loeb space) where \(\mathcal{A}_L\) is the smallest (external) sigma-algebra containing \(\mathcal{A}\).

• A useful fact: If \(E \in \mathcal{A}_L\), and \(\varepsilon > 0\) is standard, then \(\exists A_i, A_o \in \mathcal{A}\) such that \(A_i \subseteq E \subseteq A_o\) and \(\mu(A_o) - \mu(A_i) < \varepsilon\).