AN AXIOMATICS FOR NONSTANDARD SET THEORY.

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A new axiomatic system for nonstandard set theory and an axiomatic system for the theory of hyperfinite sets will be discussed in this talk. These systems were introduced in our joint paper with P. Andreev accepted [AG]. I came to these axiomatic systems in my investigations of approximations of continuous structures by finite ones. So I'll start with a discussion of this topic.

P.Vopenka was, as far as I know, the first who suggested a new approach to mathematics based on the idea that all sets are finite but some of them are so big that the induction principle fails (the well-known paradox of a pile of sand), and we see the continous objects because some elements of finite structure are indiscernible. So they may appear as the quotient structures of some finite structutres by an appropriate equiuvalence relation, (a relation of indiscernibility). This approach may be considered as a development of the Ancient Greek atomic view to the structure of the world. The formal system constructed by Vopenka - the Alternative Set Theory (AST) - is indeed very alternative to classical mathematics and it is not clear how is this theory connected with classical mathematics. It is more natural to formalize this approach in the framework of nonstandard analysis: the big finite sets for which the induction principle fails are the hyperfinite sets. To obtain the continuous structure we have to fix a σ-substructure of a hyperfinite structure (the union of a countable sequence of internal subsets) - the set of accessible or visible elements and to consider the quotient system by an appropriate π-congruence relation (the intersection of a countable sequence of internal sets) - a relation of indicernibility. For the case of a locally compact abelian (LCA) group G this approach was developed in my monograph [Gor] where some new (standard) results on approximation of the Fourier transform on LCA groups by finite Forier transform were obtained on the base of this approach to locally compact groups. In [?] this approach received a further development, in particular, a general scheme of finite dimensional approximations of operators in $L_2(G)$ was introduced and some theorems about the spectra convergence for this aproximations were proved that generalized the results of the papers [DHV] and [DVV]. There are a lot of interesting results and open problems in this area.

It seems to me that this approach may be useful for the investigation of computer simulations of continuous structures. And it is reasonable to start with investigation of finite algebraic systems that approximate the field **R** as a topological field.

It is well known that number systems implemented in working computers (usually based on representation of reals in floating point form) are neither associative nor distributive (

see for example [KN]). Specialists in computer calculations know many examples where the incorrect results of calculations are obtained by the reason of difference between the computer number system and the field of reals even in those cases when approximate methods converge. For example, in spite of the fact that the Taylor expansion of the sin-function converges on the whole line, we can never obtain the aproximate value of $\sin(x)$ for $|x| > 2\pi$ using this expansion because for big values of x the convergence is caused by the terms of the series that are less than the least nonzero computer number and thus these terms are interpreted by the computer as zero. Indeed, all calculations are based on theorems about reals and for computer numerical systems only some approximate versions of these theorems hold. How to describe these approximate versions of classical theorems? How do they depend on concrete number systems that simulate reals in computers? Do there exists computer numerical systems that have some better properties then are used now, for example, such that they are finite associative rings?

I am going now to introduce the definition of a finite algebraic system that approximate **R** and some results about these systems that were obtained in our joint work with C.W. Henson and L.Yu. Glebsky that is now in preparation for publication. The similar definition and results were obtained for an arbitrary locally compact universal algebra (an algebraic system that contains only functional symbols in its signature)

In general, a computer numerical system can be considered as a finite algebraic system $\mathcal{A} = \langle A, \sigma \rangle$, where σ is a function signature that includes two binary operations $\oplus$ and $\odot$ simulating addition and multiplication of reals and maybe finitely many other functions also simulating some continuous real functions. We assume that $A \subseteq \mathbf{R}$ and

a) for some big enough α and small enough ε
$\forall \xi \in [-\alpha, \alpha] \exists a \in A \ (|\alpha - a| < \varepsilon)$;

b) for any n-ary function f in $\mathcal{A}$ that simulates a given continuous function

$f_{\mathbf{R}}$ we have:

$$\forall a_1, \ldots, a_n \in A \cap [-\alpha, \alpha]\, (f(a_1, \ldots, a_n) \in [-\alpha, \alpha] \Longrightarrow$$
$$\Longrightarrow |f(a_1, \ldots, a_n) - f_{\mathbf{R}}(a_1, \ldots, a_n)| < \varepsilon)\,.$$

Let us call a finite algebraic system $\mathcal{A} = \langle A, \sigma \rangle$, satisfying (a) and (b) an $\langle \alpha, \varepsilon \rangle$-approximation of $\mathbf{R}$.

Now the problems discussed above may be formulated more precisely.

1. Let φ be any theorem about $\mathbf{R}$ formulated in some formal language containing the first order language L_σ. Is it possible to construct a formula $\varphi_{\alpha,\varepsilon}$ satisfying the following condition:
   $\mathbf{R} \models \varphi$ iff there exist $\alpha_0, \varepsilon_0 > 0$ such that for any $\alpha > \alpha_0,\ 0 < \varepsilon < \varepsilon_0$ and $\langle \alpha, \varepsilon \rangle$ – approximation $\mathcal{A}$ of $\mathbf{R}$ we have $\mathcal{A} \models \varphi_{\alpha,\varepsilon}$.

2. For which theorems φ of the first order theory of $\mathbf{R}$ do there exist $\langle \alpha, \varepsilon \rangle$-approximation $\mathcal{A}$ for all big enough α and small enough ε such that $\mathcal{A} \models \varphi$? For example, is it possible to approximate $\mathbf{R}$ by finite fields, finite associative rings, etc.?

The nonstandard analysis provides us with some general approach to these problems. Let a hyperfinite system $\mathcal{A} = \langle A, \sigma \rangle$ (let us assume for simplicity that σ is finite) be an $\langle \alpha, \varepsilon \rangle$-approximation of $\mathbf{R}$ for some infinite α and infinitesimal ε. This is equivalent to the conjuntion of following conditions:

1. $\forall^{st} r \in \mathbf{R} \exists a \in A (r \approx a)$;

2. for any n-ary function f in $\mathcal{A}$ that simulates a given standard continuous function $f_{\mathbf{R}}$ we have:

$$\forall a_1, \ldots, a_n \in A \cap \mathrm{ns}({}^*\mathbf{R}) \quad f(a_1, \ldots, a_n) \approx f_{\mathbf{R}}({}^\circ a_1, \ldots, {}^\circ a_n).$$

It is easy to see that if $A_b = A \cap \mathrm{ns}({}^*\mathbf{R})$ then $\approx$ is a congruence relation for $\mathcal{A}_b = \langle A_b, \sigma \rangle$ and $\mathcal{A}_b / \approx$ is isomorphic to $\langle \mathbf{R}, \sigma \rangle$. This implies the following proposition.

Proposition 1 *Let $\varphi(x_1, \ldots, x_n)$ be a first order formula of the signature σ. Denote by $\varphi_\approx$ the formula that is obtained from φ by replacement of any atomic formula $t_1 = t_2$, where t_1 and t_2 are the terms of signature σ by $t_1 \approx t_2$. Then for any $a_1, \ldots, a_n \in A_b$*

$$\mathcal{A}_b \models \varphi(a_1, \ldots, a_n) \Longleftrightarrow \mathbf{R} \models \varphi({}^\circ a_1, \ldots, {}^\circ a_n).$$

The sentence "For any hyperfinite approximation $\mathcal{A} = \langle A, \sigma \rangle$ for any standard $r_1, \ldots, r_n \in \mathbf{R}$ for any $Aa_1, \ldots, a_n \in A_b$ such that $a_1 \approx r_1, \ldots, a_n \approx r_n$ $\mathcal{A}_b \models \varphi(a_1, \ldots, a_n)$" can be formalized in E.Nelson's Internal Set Theory (IST) [**?**] and the Nelson's algorithm of translation of nonstandard sentences into standard ones can be applied to this sentence. It gives some standard theorem about approximation versions of φ that hold in $\langle \alpha, \varepsilon \rangle$-approximations for large enough α and small enough ε. Usually we obtain some immensely complicated theorems. But for the case of positive bounded formula φ we obtain a reasonble theorems similar to those of C.W.Henson for nonstandard hulls of Banach spaces [He], [He1], [HM], [HH].

We shall use the following notations:

$$\forall_y x \varphi(x) := \forall x(|x| < y \longrightarrow \varphi(x)); \qquad \exists_y x \varphi(x) := \exists x(|x| \leq y \wedge \varphi(x)).$$

Let $\varphi(y_1, \ldots y_s)$ be a prenex positive formula of the field theory signature. So φ is of the form:

$$Q_1 x_1 \ldots Q_m x_m (\varphi_1 \vee \ldots \vee \varphi_r), \tag{1}$$

where each Q_i is either existencial or universal quantifier and each φ_j is a conjunction of atomic formulas. For any $\bar{\xi} = \langle \xi_1, \ldots \xi_m \rangle \in \mathbf{R}_+^m$ denote by $\varphi[\xi](y_1, \ldots y_s)$ the formula that is obtained from φ by replacement of each $Q_i$by $Q_{i\xi_i}$. We shall call the formulas of the form $\varphi[\xi]$ positive bounded formulas. If $\varepsilon > 0$ we shall use the notation $\varphi[\xi; \varepsilon]$ for the formula that is obtained from $\varphi[\xi]$ by replacement of each atomic formula of the form $t = s$, where t and s are terms by the formula $|t - s| < \varepsilon$. If $\bar{\eta} = \langle \eta_1, \ldots \eta_m \rangle \in \mathbf{R}_+^m$ then we shall write $\bar{\eta} >_\varphi \bar{\xi}$ if $\eta_i > \xi_i$, for $Q_i = \exists$ and $\eta_i < \xi$ for $Q_i = \forall$.

Theorem 1 *Let φ be a positive sentence (1), $\bar{\xi} \in \mathbf{R}_+^n$. Then $\mathbf{R} \models \varphi[\bar{\xi}]$ iff*

$$\forall \bar{\eta} >_\varphi \bar{\xi} \exists \alpha_0, \varepsilon_0 \forall \alpha > \alpha_0, \varepsilon < \varepsilon_0 \forall \langle \alpha, \varepsilon \rangle - \textit{approximation } \mathcal{A} \ (\mathcal{A} \models \varphi[\alpha, \varepsilon]).$$

The other theorem that was obtained here concerns the problem 2.

Theorem 2 *There exist such $\alpha, \varepsilon > 0$ that no any $\langle \alpha, \varepsilon \rangle$-approximation $\mathcal{A}$ of $\mathbf{R}$ is a finite associative ring.*

For construction of approximate versions of more general then first order theorems about $\mathbf{R}$ we'll need some external sets already so we are interested in some theory of external sets that is simple enough so that there exist an algorithm similar to those of E.Nelson. One simple theory that includes the external sets is the Nonstandard Class Theory, mentioned above.

0.1 Axiomatics

The language of **NCT** contains only two nonlogical symbols: the binary membership predicate $\in$ and the unary standardness predicate St. The basic objects are called classes. Sets are defined as members of classes:

$$SetX \rightleftharpoons \exists Y\,(X \in Y)$$

and we will use small letters for sets and capital letters for arbitrary classes. The axioms can be divided into two parts. The first part consists of axioms which do not involve the standardness predicate. Those axioms represent almost full axiomatics of **NBG** except for the Separation axiom which holds here only for internal classes. The second part includes axioms describing features of the standardness predicate. Most of them can be viewed as natural analogs to **BST** principles of boundedness, idealization, standardization and transfer. The Axiom of Chromatic Classe (ACC) has no analog in **BST** axiomatics. It rather describes the specifics of **BST**'s formula-defined classes.

Extensionality:
$\forall X\ \forall Y\,(X = Y \longleftrightarrow \forall u\,(u \in X \longleftrightarrow u \in Y))$.
Pair:
$\forall u\ \forall v\ \exists x\ \forall w\,(w \in x \longleftrightarrow\ (w = u\ \vee\ w = v))$.

Union:
$\forall x\ \exists y\ \forall u\,(u \in y \longleftrightarrow\ \exists t \in x(u \in t))$.
Power Set:
$\forall x\ \exists y\ \forall u\,(u \in y \longleftrightarrow u \subseteq x)$.
Infinity:
$\exists x\,(\ \exists y \in x\ \forall u\,(u \notin y)\ \&\ \ \forall u \in x\,(u \cup \{u\} \in x))$.
Choice:
$\forall x\,\Big(x \neq \emptyset\ \&\ \ \forall u \in x\,(u \neq \emptyset) \longrightarrow$
$\longrightarrow \exists f\,(Fncf\ \&\ \forall u \in x\ \exists v\,(\langle u, v\rangle \in f\ \&\ v \in u))\Big)$. **Replacement:**
$\forall F\ \forall x\ \exists y\ \forall a\ \forall b\,(FncF\ \&\ \langle a, b\rangle \in F\ \&\ a \in x \longrightarrow b \in y)$.
Regularity:
$\forall x\ \exists u \in x\,(u \cap x = \emptyset)$.
A formula of the language of **NCT** is called normal, if only set variables are quantified in it.

Existence of classes:
$\forall X_1, \ldots\ \forall X_n\ \exists X\ (X = \{x : \phi(x, X_1, \ldots, X_n)\})$

for any normal formula ϕ.

Notations: U ={x — x=x } - the class of all sets ;
S ={x — St(x)} - the classs of all standard sets ;
$\forall^{st} X\ \phi(X) \longleftrightarrow \forall X(\mathrm{St}(X) \longrightarrow \phi(X))$;
$\exists^{st} X\ \phi(X) \longleftrightarrow \exists X(\mathrm{St}(X)\ \&\ \phi(X))$;
for any **NCT**-formula ϕ.

The quantifiers $\forall^{st}$ and $\exists^{st}$ are called external.

Separation:
$\forall^{st} X\ \forall^{st} x\,(Set(X \cap x))$.

Boundedness:
$\forall x\ \exists^{st} z\,(x \in z)$.

Transfer:
$\forall^{st} X\,(\exists x\,(x \in X) \longrightarrow \exists^{st} x\,(x \in X))$.

Formulae with no occurences of the standardness predicate are called internal. **Existence of standard classes:**
$\forall^{st} X_1, \ldots\ \forall^{st} X_n\ \exists^{st} X\,(X = \{x : \phi(x, X_1, \ldots, X_n)\})$
for any internal normal formula ϕ.

This axiom implies for example that the classes U and $\in = \{\langle x, y\rangle \mid x \in y\}$ are standard.

For any class X we shall denote by ${}^{\circ}X$ the class of all its standard elements:

$${}^{\circ}X = \{x : x \in X\ \&\ \mathrm{St}x\}.$$

Notation:

$$\forall^{\mathrm{st\ fin}} c\ \phi(c) \longleftrightarrow \forall c(\mathrm{St}(c)\ \&\ (c \text{ is finite}) \longrightarrow \phi(c))$$

Idealization:
$\forall x\ \forall^{st} c_0\,(\forall^{\mathrm{st\ fin}} c \subseteq c_0\ \exists y \in x\,(c \subseteq y) \longleftrightarrow \exists y \in x\,({}^{\circ}c_0 \subseteq y))$.

Standardization:
$\forall X\ \exists^{st} Y\,({}^{\circ}Y = {}^{\circ}X)$.

We will use Nelson's notation ${}^{\mathrm{s}}X$ for the standard class containing the same standard elements as X.

Note that the standardization axiom is accepted for arbitrary classes, not just for semisets.

A class X is called standard relative to a set p or p–standard iff it can be represented as a "cut" of some standard class Y by the set p : $\mathrm{St}_p X \rightleftharpoons$

$\exists^{st} Y\,(X = Y \mid p)$, where
$Y \mid p = \{x : \langle p, x\rangle \in Y\}$. A class is called internal iff it is p–standard for some set p:

$$Int X \rightleftharpoons \exists p\,(st_p X).$$

The intersection of all p–standard sets containing a given set x is called p–monad of the set x:

$$\mu_p(\,x\,) = \{y : \forall^{st_p} a\,(x \in a \longrightarrow y \in a)\}.$$

Axiom of Chromatic Classes (ACC):
$\forall X\ \exists p\ \forall x \in X\,(\mu_p(\,x\,) \subseteq X)$.

Remark It can be proved in exactly the same way as for **NBG** that the axiom scheme of existence of classes is equivalent to finitely many axioms. The same holds also for the axiom scheme of existence of standard classes. Thus **NCT** is finitely axiomatizable.

In [AG] the Theory of Hyperfinite Sets (THS) was also introduced. It is obtained from NCT by replacement of the Axiom of Infinity by its negation and the ACC by the Separation Principle. The subclasses of V_ω form a model of THS and thus THS is consistent. It is easy to see that all the properties of the hyperfinte structures that approximate the continuous structures may be formulated in THS. The following problem seems to be very important.

Which theorems of NCT involving only the hyperfnite sets and the classes of hyperfinite sets are provable in THS?

Let us consider some examples concerning this problem.

It can be proved [Gor] that a locally compact separable group G is approximable by finite groups iff there exits a hyperfinite group G, a σ-subgroup G_b (the union of a countable sequence of internal sets) of G, and a normal π-subgroup G_0 (the intersection of a countable sequence of internal sets) of G_b such that G is isomorphic to $G^{\#} = G_b/G_0$. Moreover, G is homeomorphic to $G^{\#}$ with respect to the topology on $G^{\#}$ defined by the uniformity on it that consits of all the images of all internal subsets of G_b^2 that contain the set $\{\langle u, v\rangle \mid u \cdot v^{-1} \in G_0\}$ by the canonical map. It can be proved that the topology is locally compact iff (LC):

for any internal sets U, F such that $G_0 \subset U \subset G_b$ and $F \subset G_b$ there exist standardly finite set $K \subset F$ such that $U \cdot K \supset F$. $\langle G, G_b, G_0\rangle$ Since all locally compact abelian groups are approximable by finite ones any theorem about these groups may be formulated as some theorem about the triples of the form $\langle G, G_b, G_0\rangle$ that satisfies (LC). Most part of these theorems can be proved in NCT.

Which of them can be proved in THS?

Let us consider an example. Let a triple $\langle G, G_b, G_0\rangle$ defines an LCA group, $H = \widehat{G}$ is the internal group of characters of G, $H_b = \{\chi \in \widehat{G} \mid \chi|G_0 \approx 1\}$ - the external σ-subgroup of S-continuous characters of G, $H_b = \{\chi \in \widehat{G} \mid \chi|G_0 \approx 1\}$ - the external *pi*-subgroup of characters, approximately equal to the unit character. It is easy to understand that $H^{\#}$ is a subgroup of $\widehat{G}$. It was proved [Gor] that for all groups with compact open subgroup

$$H^{\#} = \widehat{G} \tag{1}$$

.

The equality (1) holds also for all known approximations of the additive group $\mathbf{R}$. According do the structural theory of LCA group any LCA group is a direct sum of $\mathbf{R^n}$ and a group with a compact open subgroup. So any LCA group can be constructed from a triple $\langle G, G_b, G_0\rangle$ that satisfies (1). But does (1) hold for any triple $\langle G, G_b, G_0\rangle$ that satisfies (LC). This is an open question. This question is equivalent to the following one [Gor]. Consider $G'_b = \{\in G \mid \forall h \in H_0 h(b) \approx 1\}$ and $G'_0 = \{\in G \mid \forall h \in H_b h(b) \approx 1\}$. It is easy to see that $G_0 \subseteq G'_0$, $G_b \subseteq G'_b$. It is proved in [Gor] that $G_0 = G'_0$. The equality

$$G_b = G'_b \tag{2}$$

is equivalent to (1). It can be also considered as a hyperfinite analog of Pontrjagin duality principle, which can be deduced from (2). But even in the case when G contains a compact open subgroup and thus, the equality (2) holds, it is proved with the help of Pontrjagin's duality principle. It is easy to see that (2) can be formulated in the language of NCT and only the hyperfinite sets and their subclusses are involed in this formula. It is interesting to understand if (2) can be proved in the theory THS.

Let us discuss a particular case when the group $G_{\#}$ is compact.

It was proved that a triple $\langle G, G_b, G_0\rangle$ is such that $G^{\#}$ is compact iff (LC) holds and $G_b = G$. If $\varphi : G \to {}^*\mathbf{C}$ is an internal function, such that $\forall a, b \in G(a - b \in G_0 \longrightarrow \varphi(a) \approx \varphi(b)$ (we say in this case that φ is S-continuous) then φ defines a continuous function $\widetilde{\varphi} : G^{\#} \to \mathbf{C}$ such that $\widetilde{\varphi}(a^{\#}) = {}^{\circ}\varphi(a)$, where $a^{\#} \in G^{\#}$ is the image of $a \in G$ by the canonical map. The equalities (1) and (2) are equivalent to the fact that any character $\chi \in \widehat{G_\#}$ is of the form $\widetilde{h}$ for an appropriate S-continuous internal character h. First of all it was proved that every continuous function $f : G^\# \to \mathbf{C}$ is of the form $\widetilde{\varphi}$ for an appropriate internal S-continuous φ. This theorem is equivalent to a following proposition in the language of NCT:

For any (may be external) S-continuous map $F : G \to \mathbf{C}$ there exists an internal S-continuous map $\varphi : G \to \mathbf{C}$ such that $\forall a \in G \ \varphi(a) \approx F(a)$.

Using the hyperfinite approximation of the additive group $\mathbf{C}$ we can easily formalute this proposition in the language of THS (i.e. in a such way that only hyperfinite sets would be involved).

The proof of this theorem that contains in [Gor] is based on the Stone-Weierstrass theorem applied to $\mathbf{C}(G^\#)$ - we show that the algebra of functions of the form $\widetilde{\varphi}$ satisfies the conditions of this theorem. This proof can not be formalized in NCT but not in THS. But one of the important parts of it this proof - the proof of the analog of Urison's lemma for internal S-continuous functions that can be easily formalized in THS. To formalize the whole proof in THS we have to find some hyperfinite analog of Stone - Weierstrass theorem provable in THS.

It would be also interesting to construct the approximating versions of these theorems as it was done for positive bounded formulas.

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