ZFCΩ: A NONSTANDARD SET THEORY WHERE ALL SETS HAVE NONSTANDARD EXTENSIONS

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Foundations of Nonstandard Methods

AXIOMATIC SET THEORIES: Include nonstandard methods within a unified general axiomatic system for mathematics.

ELEMENTARY PRESENTATIONS: Provide "elementary" introductions to nonstandard analysis which avoid (as far as possible) technical notions from mathematical logic.

Superstructure Approach

- A *standard universe*, namely a superstructure $V(X) = \cup_{n \in N} V_n(X)$ where $V_0(X) = X$ and $V_{n+1}(X) = \mathcal{P}(V_n(X))$. X contains a copy of $\mathbf{N}$.
- A *nonstandard universe* $V(Y)$.
- A *nonstandard embedding*

$$* : V(X) \to V(Y)$$

(*i*) ${}^*X = Y$
(typically ${}^*x = x$ for all $x \in X$)
(*ii*) *Transfer principle*:
$*$ preserves all "elementary" properties.
(*iii*) $\mathbf{N}$ is a proper subset of ${}^*\mathbf{N}$.

In the nonstandard universe:

- x is **standard** if $x = {}^*y$ for some y.
- x is **internal** if $x \in {}^*y$ for some y.
- x is **external** if x is NOT internal.

By *transfer*:

"Every (nonempty) bounded subset of $\mathbf{N}$ has a greatest element" $\Longleftrightarrow$ "Every (nonempty) $A \in {}^*\mathcal{P}(\mathbf{N})$ has a greatest element".

Hence $\mathbf{N}$ is *external*.

- κ-**saturation** (κ an infinite cardinal): If $|\mathcal{F}| < \kappa$ is a family of internal sets with the finite intersection property FIP (i.e. $\cap\mathcal{F}' \neq \emptyset$ for all finite $\mathcal{F}' \subseteq \mathcal{F}$) then $\cap\mathcal{F} \neq \emptyset$.

Why only the *finite* levels in the hierarchy ?
If ω is in the standard model, then ${}^*\omega$ is NOT wellfounded.

Why $*$ is only a *bounded* elementary embedding ?
Otherwise, transfer the following:
"$\forall x \in \mathbf{N}\, \exists A_x\, \exists f : [0, x] \to \langle A_x, \in\rangle$ isomorphism of ordered structures". For any infinite $\xi \in {}^*\mathbf{N}$, the corresponding A_ξ is NOT wellfounded.

Give up the regularity axiom.

Limitations of the Superstructure Approach

- In principle, nonstandard methods are NOT related to the notion of cumulative hierarchy.
- Different superstructures are needed for different problems.
- Superstructures only model a fragment of ZFC.
- Transfer is formulated by using the technical notion of bounded elementary embedding.
- Only a limited amount of saturation is available

NONSTANDARD SET THEORIES: Axiomatize nonstandard methods in the full generality of set theory.

Internal viewpoint

Independently introduced by Karel Hrbàček and Edward Nelson in the 70's.

- Use the *standardness* predicate `st` and the *internalness* predicate `int`.

Infinitesimals and other idealized objects are already there.

$$\mathbf{S} = \{x : \mathtt{st}(x)\}; \quad \mathbf{I} = \{x : \mathtt{int}(x)\}$$

S is a proper subset of **I**.

- **TRANSFER** For each $\in$-formula φ and $x_1, \ldots, x_n \in \mathbf{S}$:

$$\varphi^S(x_1, \ldots, x_n) \Longleftrightarrow \varphi^I(x_1, \ldots, x_n)$$

- **SATURATION** (Idealization)

External viewpoint

Original view of Abraham Robinson.

$$* : \mathbf{V} \to \mathbf{M}$$

- Every object A in the *standard universe* $\mathbf{V}$ has a "nonstandard" extension ${}^*A \in \mathbf{M}$ in the *nonstandard universe* $\mathbf{M}$:

$$\mathbf{S} = \{{}^*x : x \in \mathbf{V}\}$$

$$\mathbf{I} = \{y : \exists x \in \mathbf{V} \text{ with } y \in {}^*x\}$$

$\mathbf{S}$ is a proper subset of $\mathbf{I}$.

- **TRANSFER** For each (bounded) $\in$-formula φ and $x_1, \ldots, x_n \in \mathbf{V}$:

  $$\varphi^V(x_1, \ldots, x_n) \Longleftrightarrow \varphi^M({}^*x_1, \ldots, {}^*x_n)$$

- **κ-SATURATION** (for a suitable κ)

What properties a nonstandard set theory NST should satisfy?

- *All* principles of mathematics ZFC$^-$ (without regularity).
- *Transfer principle* for all sets of ordinary mathematics (**TP**).
- Unlimited levels of *saturation* (**US**).

HRBÀČEK's paradox.

ZFC$^-$ + **TP** + **US** $\Longrightarrow$
$^*A$ is a proper class for each infinite A.

Proof. Consider the family of internal sets $\mathcal{F} = \{{}^*A \setminus \{\xi\} : \xi \in A\}$ and apply saturation.

Nelson's Internal Set Theory IST

The paradox is avoided because *all* sets are internal. There is no external notion of cardinality.

Hrbàček's NS_1 and NS_2

- **TP** + **US** + ZFC$^-$ *except* powerset and choice, is a conservative extension of ZFC.
- **TP** + **US** + ZFC$^-$ *except* replacement, is a conservative extension of ZFC.

Motivations in formulating NST

"FORMALISTIC" VIEW:
Nonstandard methods are "short-cuts" for standard methods.

"REALISTIC" VIEW:
Look for axioms which are true in the "real" world.

- In the "real" world, ZFC^- holds, with separation and collection for *all* formulas in the extended language.

E.g. **standardization** should be true:
$\forall X \;\; {}^oX = \{x \in X : \exists y \; x = {}^*y\}$ is a set

- In the "real" world, unlimited levels of saturation are available.

Research in *nonstandard set theories* can be seen as an attempt to answer the following question:

How to avoid Hrbàček's paradox?

A (partial) list of proposals:

Nelson's *Internal Set Theory* IST;
Hrbàček's *Nonstandard Set Theories* NS_1, NS_2 and NS_3;
Kawai's *Nonstandard Set Theory* NST;
Fletcher's *Stratified Nonstandard Set Theory* SNST;
Ballard's *Enlargement Set Theory* EST;
Zermelo-Fraenkel-Boffa ZFBC, proposed by Ballard and Hrbàcček;
Kanovei-Reeken's *Bounded Set Theory* BST and *Hrbàček Set Theory* HST;
Péraire's *Relative Set Theory* RST;
Di Nasso's *ZFC;
Gordon-Andreyev *Nonstandard Class Theory* NCT;
etc. ....

***ZFC**

- ZFC^-, where separation and replacement are assumed for *all* $\in$-$*$-formulas ($*$ is a new symbol).
- The nonstandard embedding $* : \mathbf{S} \to \mathbf{I}$ is defined for *all* wellfounded sets, that is the standard universe is $\mathbf{S} = \mathbf{WF}$.
- κ-saturation for *all* $\in$-definable cardinals.

Theorem.
Let φ be any $\in$-formula. Assume that κ-saturation $\Rightarrow$ $\varphi^S(x)$ for all $x \in \mathbf{S}$ with $|x| < \kappa$. Then $\varphi^S(x)$ for all $x \in \mathbf{S}$.

Proof. By contradiction, let

$$\kappa = \min\{|x| : x \in \mathbf{S} \text{ and } \neg\varphi^S(x)\}$$

κ is $\in$-definable, thus κ^+-saturation implies $\varphi^S(x)$ for all standard $|x| \leq \kappa$, a contradiction. $\dashv$

Example. A typical proof in nonstandard analysis shows that:
"If $|X|^+$-saturation holds and X is the topological product of a family of compact spaces, then X is compact".
As a consequence, *ZFC proves Tychonov's theorem for *all* topological products X.

Theorem.
Every model $A \models ZFC$ has an elementary end extension $A \prec_e A'$ such that $A' = (\mathbf{WF})^B$ for some $B \models {}^*ZFC$.

Corollary. *For any $\in$-sentence σ:*

$$ZFC \vdash \sigma \;\Leftrightarrow\; {}^*ZFC \vdash \sigma^{WF}$$

We want more:

- Nonstandard methods should be applied to *all* objects of mathematics (not only to the standard ones).

- κ-saturation should be available for each fixed κ.

- A presentation to nonstandard analysis should be possible in truly "elementary" terms.

ZFCΩ

- ZFC$^-$ where separation and collection are assumed for *all* $\in$-Ω-formulas (Ω is a new symbol).
- Ω is a function defined on the class of sequences, i.e. on the class of those functions f whose domain is an infinite cardinal κ.

For κ-sequences, denote by $\Omega(f) = f(\alpha_\kappa)$.

- If f is a sequence of nonempty sets:

$f(\alpha_\kappa) = \{g(\alpha_k) : g(i) \in f(i) \text{ for all } i\}$

- Point-wise unions and intersections are preserved by Ω, i.e.
  1. $f(i) = g(i) \cup h(i)$ for all $i \Rightarrow f(\alpha_\kappa) = g(\alpha_\kappa) \cup h(\alpha_\kappa)$;
  2. $f(i) = g(i) \cap h(i)$ for all $i \Rightarrow f(\alpha_\kappa) = g(\alpha_\kappa) \cap h(\alpha_\kappa)$.
- If $f(\alpha_\kappa) = g(\alpha_\kappa)$ then $(\varphi \circ f)(\alpha_\kappa) = (\varphi \circ g)(\alpha_\kappa)$ (provided such compositions are defined)

Call κ-**internal** any element of the form $f(\alpha_\kappa)$, and denote by $\mathbf{I}_\kappa$ the collection of κ-internal sets.

- *κ-saturation:*
  If $|\mathcal{F}| \leq \kappa$ is a family of κ-*internal* sets with the FIP, then $\cap\mathcal{F} \neq \emptyset$.

Define $J_\kappa(A) = \{f(\alpha_\kappa) : f : \kappa \to A\}$, the *$\kappa$-nonstandard embedding.*

Theorem.
For each κ, $J_\kappa : \mathbf{V} \to \mathbf{I}_\kappa$ is a κ^+-saturated bounded elementary embedding of the universal class of all sets $\mathbf{V}$.

If we also assume the following *minimality axiom*, then J_κ is *fully* elementary.

- *Minimality Axiom*
  Every set is obtained from the emptyset by transfinite iterations of the powerset and nonstandard extension operations. That is

$$\mathbf{V} = \cup_\alpha W_\alpha \text{ where}$$

$$W_0 = \emptyset;\ W_{\alpha+1} = \mathcal{P}(\cup_{\kappa\le\alpha} J_\kappa(W_\alpha) \cup W_\alpha);$$
$$W_\lambda = \cup_{\alpha<\lambda} W_\alpha \text{ if } \lambda \text{ limit.}$$

Theorem.
Every model $A \models ZFC$ is the wellfounded part $A = \mathbf{WF}^B$ of some $B \models ZFC\Omega$.

Corollary. *For any $\in$-sentence σ:*

$$\mathrm{ZFC} \vdash \sigma \;\Leftrightarrow\; \mathrm{ZFC}\Omega \vdash \sigma^{WF}$$

If we reduce to countable sequences, we get the *Alpha-Theory*, a truly "elementary" approach to nonstandard analysis which, in principle, can even be suitable at the freshman level.

References:

M. Di Nasso, *On the foundations of nonstandard mathematics*, Mathematica Japonica, vol.50, 1999, pp. 131-160.
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