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# NONSTANDARD SET THEORY AND SOME OF ITS PROBLEMS.

KAREL HRBACEK

## What is nonstandard set theory?

AST, BST, BNST, EST, HST, IST, NCT, NST, NS<sub>1</sub>, NS<sub>2</sub>, NS<sub>3</sub>, RIST, SNST,  
\*ZFC,....??

[ See M. Di Nasso, *On the foundations of nonstandard mathematics*, *Mathematica Japonica* 50 (1999), pp. 131 - 160 for a survey.]

## Core theory:

The primitive concepts are:  
variables  $x, y, \dots$   
binary predicate  $\in$   
unary predicates  $\mathbb{S}$  (*standard*) and  $\mathbb{I}$  (*internal*).

*Classes* will be used informally to denote extensions of formulas; in particular:  
 $\mathbb{S} := \{ x \mid \mathbb{S}(x) \}$ ,  $\mathbb{I} := \{ x \mid \mathbb{I}(x) \}$ .

If  $\phi$  is an  $\in$ -formula,  $\phi^{\mathbb{S}}$  is the formula obtained from  $\phi$  by replacing each subformula of the form  $(\exists x) \psi$  [  $(\forall x) \psi$ , resp.] by  $(\exists x) (\mathbb{S}(x) \wedge \psi) \equiv (\exists x \in \mathbb{S}) \psi$  [  $(\forall x) (\mathbb{S}(x) \Rightarrow \psi) \equiv (\forall x \in \mathbb{S}) \psi$ , resp.]. Similar conventions will be used for  $\mathbb{I}$  and other classes. Also, if  $R$  ( $c, F$ , resp.) is a relation (constant, operation, resp.) defined by  $\phi$ , then  $R^{\mathbb{S}}$  ( $c^{\mathbb{S}}, F^{\mathbb{S}}$ , resp.) denotes the relation (constant, operation, resp.) defined by  $\phi^{\mathbb{S}}$  (provided that the appropriate existence and uniqueness statements hold), and similarly for other classes.

Occasionally it will be convenient to write  $\mathbb{S} \models \phi$  in place of  $\phi^{\mathbb{S}}$  and say that  $\mathbb{S}$  *satisfies*  $\phi$ , etc.

**Core axioms:**

0.  $x \in \mathbb{S} \Rightarrow x \in \mathbb{I}$ .
- I.  $\phi^{\mathbb{S}}$  where  $\phi$  is any axiom of ZFC.
- II.  $x_1 \in \mathbb{S} \wedge \dots \wedge x_n \in \mathbb{S} \Rightarrow (\phi^{\mathbb{S}}(x_1, \dots, x_n) \equiv \phi^{\mathbb{I}}(x_1, \dots, x_n))$   
 where  $\phi$  is any  $\in$ -formula whose free variables are among  $x_1, \dots, x_n$ .  
 (*Transfer.*)  
 [In particular,  $\phi^{\mathbb{I}}$  holds for all axioms  $\phi$  of ZFC.]
- III.  $(\exists x \in \omega^{\mathbb{S}}) (x \notin \mathbb{S})$ .  
 (*A very weak form of Idealization.*)
- IV.  $x \in \mathbb{I} \wedge y \in x \Rightarrow y \in \mathbb{I}$ .  
 (*Transitivity of  $\mathbb{I}$ .*)
- V.  $(\forall x) (\exists y \in \mathbb{S}) (\forall z \in \mathbb{S}) (z \in x \equiv z \in y)$ .  
 (*Standardization .*)
- VI. Extensionality, Pairing, Union, Separation, and Standard Size Replacement.

[The axiom schemata of Separation and Replacement allow arbitrary  $\in$ - $\mathbb{S}$ - $\mathbb{I}$ -formulas. A set  $X$  is of *standard size* if there is a standard set  $A$  and a 1-1 mapping  $F$  of  $A \cap \mathbb{S}$  onto  $X$ .]

**Some simple consequences:**

$(\forall y \in \mathbb{I}) (\exists x \in \mathbb{S}) (y \in x)$  (*Boundedness.*)

Transitivity of  $\mathbb{I}$  implies that many basic set-theoretic formulas and operations are *absolute* for  $\mathbb{I}$ ; for example,  
 $x_1 \in \mathbb{I} \wedge \dots \wedge x_n \in \mathbb{I} \Rightarrow (\phi^{\mathbb{I}}(x_1, \dots, x_n) \equiv \phi(x_1, \dots, x_n))$  holds for all restricted formulas, and  $(\forall x, y \in \mathbb{I}) (\mathcal{F}_i^{\mathbb{I}}(x, y) = \mathcal{F}_i(x, y))$  holds for all Gödel operations.

Standardization postulates that for any set  $x$  there is a standard set  $y$  having the same standard elements as  $x$ ; by Extensionality in the standard universe, this  $y$  is unique. We denote it by  $x^\circ$ .

Standard Size Replacement allows the construction of the von Neumann cumulative hierarchy, and an isomorphism  $*$  between  $(\mathbb{V}, \in)$  and  $(\mathbb{S}, \in)$ .

[ $\mathbb{V} := \cup_{\alpha \in \mathbb{O}_n} V_\alpha$  is the class of all *well-founded sets*.]

Thus one can choose to work either in the *internal picture* just described, or the equivalent *external picture* (familiar from superstructures).

**Core axioms are not adequate**

for the practice of Nonstandard Analysis. One needs in addition:

*Stronger Idealization* (Axiom III.)

*Stronger external set theory* (Axiom VI.).

**Definition.** A collection of sets  $\mathcal{F}$  has the *finite intersection property* (FIP) if  $\bigcap \mathcal{F}_0 \neq \emptyset$  for every finite  $\mathcal{F}_0 \subseteq \mathcal{F}$ .

**Definition.**  $\kappa$ -*enlargement property* ( $\kappa$ -*saturation property*, resp.) asserts: If  $\mathcal{F}$  is a collection of standard sets (internal sets, resp.) with FIP and  $|\mathcal{F}| < \kappa$  then  $\bigcap \mathcal{F} \neq \emptyset$ .

*Enlargement*  $\equiv (\forall \kappa)$   $\kappa$ -enlargement property;

*Saturation*  $\equiv (\forall \kappa)$   $\kappa$ -saturation property.

“Naively,” we would like to postulate Saturation. However:

**Proposition 1:**

“*Core Axioms + Enlargement + Replacement + PowerSet*” is inconsistent;

“*Core Axioms + Enlargement + Replacement + Well-Ordering*” is inconsistent.

**Proposition 2:**

*Core Axioms + Saturation + Replacement (+Standard-Size Choice + Dependent Choice + Well-Foundedness over  $\mathbb{I}$ )* is a conservative extension of ZFC.

This is the theory HST of Kanovei - Reeken.

They showed that even HST + *Isomorphism Property* of Henson is a conservative extension of ZFC. In HST + *Isomorphism Property* there is a unique internal model of  $\text{Th}(\mathbb{R})$ , up to external isomorphism.

**Proposition 3:**

“*Core Axioms + Saturation + Power set + Well-Ordering*” is consistent, but is NOT a conservative extension of ZFC. In this theory one can prove existence of a class  $\mathbb{L}[\mathbb{I}]$  such that  $(\mathbb{L}[\mathbb{I}], \mathbb{I}, \mathbb{S}, \in) \models \text{HST}$ .

Lack of full  $\text{ZFC}^-$  in the external universe makes many nonstandard constructions awkward or impossible. It is clear one can have  $\text{ZFC}^-$  in the external universe, if Saturation is replaced by  $\kappa$ -saturation for a fixed (definable)  $\kappa$ . However, this means using different theories for different problems. Fletcher had the idea of axiomatizing a *single* cosmos that allows such universes for all  $\kappa$ .

**Classes that satisfy Transfer:**

**Definition:** A class  $\mathbb{U}$  satisfies *Transfer* if  $\mathbb{S} \subseteq \mathbb{U} \subseteq \mathbb{I}$  and  $(\mathbb{U}, \mathbb{S}, \in) \models \text{Transfer}$ .  
It then follows that also  $(\mathbb{I}, \mathbb{U}, \in) \models \text{Transfer}$ .

**Definition.** Let  $\kappa$  be an infinite cardinal. An internal set  $x$  is  $\kappa$ -constrained if  $x \in {}^*a$  for some  $a \in \mathbb{V}$  with  $|a| \leq \kappa$ .  
Set  $\mathbb{C}_\kappa := \{x \in \mathbb{I} \mid x \text{ is } \kappa\text{-constrained}\}$ .

**Proposition 4:**  $\mathbb{C}_\kappa$  satisfies Transfer and  $\kappa^+$ -saturation.

**Proposition 5:** There is a class  $\mathbb{H}_\kappa$  such that  
 $(\mathbb{H}_\kappa, \mathbb{C}_\kappa, \mathbb{S}, \in) \models \text{HST}_\kappa + \text{Power set}$ .

[HST $_\kappa$  is HST where Saturation has been replaced by  $\kappa^+$ -saturation.]

For any class  $\mathcal{X} \subseteq \mathbb{I}$  we define the class of *sets standard relative to  $\mathcal{X}$* :  
 $\mathbb{S}[\mathcal{X}] := \{f(x_1, \dots, x_n) \mid f \text{ is a standard function, } x_1, \dots, x_n \in \mathcal{X}\}$ .

**Proposition 6:**

$\mathbb{S}[\mathcal{X}]$  satisfies Transfer.

It is the smallest class that satisfies Transfer and contains  $\mathcal{X}$ .

$\mathbb{C}_\kappa = \mathbb{S}[{}^*\kappa]$ .

**Proposition 7:** For all  $\kappa$ , there are classes  $\mathbb{J}_\kappa$  and  $\mathbb{H}'_\kappa$  such that  
 $(\mathbb{H}'_\kappa, \mathbb{J}_\kappa, \mathbb{S}, \in) \models \text{HST}_\kappa + \text{Power set} + \text{Choice}$ .

The theory “HST $_\kappa$ +Power set+ Choice” is equivalent to  
“Core Axioms + ZFC $^-$  +  $\kappa^+$ -saturation”.

### A single universe after all?

Di Nasso: *Definable Saturation* [ $\kappa$ -saturation for all definable  $\kappa$ ].

\*ZFC is essentially “Core Axioms+ ZFC<sup>-</sup> + Definable Saturation.”

Fact: any standard statement about a standard set  $x$  that is provable in \*ZFC under the assumption of  $\kappa$ -saturation is provable in \*ZFC (hence, in ZFC) alone, provided only that the level of saturation that is needed for the proof is  $\in$ -definable in terms of  $x$  (as is always the case in practice).

For many other purposes (such as, the construction of Loeb measures) Definable Saturation is also sufficient.

In another direction, I have considered BNST = Core Axioms+ ZFC<sup>-</sup> with an additional axiom (in lieu of  $\kappa$ -saturation):

**Axiom A<sub>1</sub>**: Every standard ordered field has infinitesimals.

This statements is equivalent to:

For every infinite regular  $\kappa$  there exist internal sets that are *strictly  $\kappa$ -constrained*, i.e.,  $\kappa$ -constrained, but not  $\lambda$ -constrained for any  $\lambda < \kappa$ .

A weaker statement is:

**Axiom A<sub>0</sub>**: For every infinite cardinal  $\lambda$  there exist internal sets that are not  $\lambda$ -constrained.

**Proposition 8**: *Con (BNST + A<sub>0</sub>) implies Con (ZFC + there exists a measurable cardinal).*

**Proposition 9**: *Con (ZFC + there exists a proper class of measurable cardinals) implies Con (BNST + A<sub>0</sub>).*

**Axiom B<sub>1</sub>**: For every infinite  $\kappa$  there exist internal sets that are strictly  $\kappa$ -constrained.

**Definition.** A collection of sets  $\mathcal{F}$  has the  *$\kappa$ -intersection property* ( $\kappa$ -IP) if  $\bigcap \mathcal{F}' \neq \emptyset$  for every  $\mathcal{F}' \subseteq \mathcal{F}$  with  $|\mathcal{F}'| < \kappa$ .

**Axiom A<sub>2</sub>**: For every regular  $\kappa$  and every collection  $\mathcal{F}$  of standard subsets of  ${}^*\kappa$ , if  $\mathcal{F}$  has the  $\kappa$ -IP then  $\bigcap \mathcal{F} \neq \emptyset$ .

Relative consistency of the stronger versions of A and B is at present unknown. Work of Foreman, Magidor and Shelah shows that nonregular ultrafilters are consistent relative to “ZFC + huge cardinals exist.” Axiom A<sub>1</sub> requires existence of many very highly nonregular ultrafilters.

**Slim universes.**

**Definition.** The internal universe is *selective* if for any  $\mu, \nu \in {}^*\omega \setminus \omega$  there is a standard  $f$  such that  $f(\nu) = \mu$ .

**Definition.** A standard ultrafilter  $\mathcal{U}$  is *realized* by  $\xi \in \mathbb{I}$  if  $X \in \mathcal{U} \equiv \xi \in X$  holds for all standard  $X$ .

If the internal universe is selective, every realized nonprincipal ultrafilter on  $\omega$  is selective (=Ramsey), and all realized nonprincipal ultrafilters on  $\omega$  are isomorphic.

Is “BNST + The internal universe is selective+ $A_0$ ” consistent?

YES (if a proper class of measurable cardinals is consistent).

What about Con( BNST + The internal universe is selective+ $A_1$ )?

This implies existence of a uniform ultrafilter  $\mathcal{U}$  on  $\omega_1$  such that for every  $f : \omega_1 \rightarrow \omega$  the ultrafilter  $f_*(\mathcal{U})$  is either principal or selective.

**Axiom  $S_\kappa$**  : If  $\xi, \eta$  are strictly  $\kappa$ -constrained then there is a standard  $f$  such that  $f(\xi) = \eta$ .

Question: Con(BNST +  $(\forall \kappa)S_\kappa$  +  $A_1$ )?

Above I considered “relativizing” the internal universe of NST. Gordon and Peraire were the first to consider

**Relativizing the standard universe.**

**Proposition 10:** If  $\mathbb{S} \neq \widehat{\mathbb{S}}$  and  $\widehat{\mathbb{S}} \cap {}^*\omega \neq {}^*\omega$  then  $(\mathbb{I}, \widehat{\mathbb{S}}, \epsilon)$  fails to satisfy Standardization for internal sets.

**Proposition 11:** (HST)

If  $\mathbb{S} \neq \widehat{\mathbb{S}}$  and  $(\mathbb{I}, \widehat{\mathbb{S}}, \epsilon) \models \text{Enlarging}$  then for any  $\kappa$  there is  $\widehat{\mathbb{I}}$  such that  $(\widehat{\mathbb{I}}, \widehat{\mathbb{S}}, \epsilon) \models \text{Standardization} + \kappa^+$ -saturation.

If  $\widehat{\mathbb{S}}$  is *thin* (ie.,  $\widehat{\mathbb{S}} \cap X$  is of standard size, for any  $X \in \widehat{\mathbb{S}}$ ) then there is an external universe  $\widehat{\mathbb{H}}$  so that  $(\widehat{\mathbb{H}}, \widehat{\mathbb{I}}, \widehat{\mathbb{S}}, \epsilon) \models \text{HST}_\kappa$ .

If one wants to have full Idealization, then only consistency results can be expected. We need to talk about classes that are not definable in order to formulate them.

Let HCT be to HST what Godel-Bernays is to ZFC.

The Relativization Property: For any internal  $\xi$  there is a universe  $\widehat{\mathbb{S}}$  such that  $\xi \in \widehat{\mathbb{S}}$  and  $(\mathbb{I}, \widehat{\mathbb{S}}, \epsilon) \models \text{Standardization} + \text{Idealization}$ .

**Proposition 12** (?)

HCT + Relativization Property is consistent.

The “internal part” of this result is essentially due to Peraire.

There are also applications of Nonstandard Analysis where an external universe of one enlargement is taken as the standard universe of another enlargement (see eg. Molchanov). The first, and so far the only axiomatic set theory that allows this kind of relativization of the standard universe is Ballard’s EST. But it seems that the idea can also be implemented in HST, provided that the axiom of Well-Foundedness over  $\mathbb{I}$  is replaced by a suitable anti-foundation (over  $\mathbb{I}$ ) axiom.