PICKING ONE POINT IN EACH MONAD COUNTABLY DETERMINEDLY

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Notation for Loeb Space

Let $H \in {}^*\mathbb{N} \setminus \mathbb{N}$, $\Omega = \{0,1,\ldots, H\text{-}1\}$ and μ_L be the Loeb measure generated by a normalized uniform counting measure on all internal subsets of Ω.

Which sets are not Loeb measurable?

An infinite initial segment U of Ω is called a cut if $U + U \subseteq U$.

For each $x \in \Omega$, the U-monad of x is $M_U(x) = \{y \in \Omega : |y - x| \in U\}$.

A set C $\subseteq \Omega$ is called a U-choice set if for every $x \in \Omega$, $|C \cap M_U(x)| = 1$.

Fact:

Let $U \subseteq \Omega$ be any cut. If C is a U-choice set, then $\mu_L\ (C) = 0$ or C is not Loeb measurable.

Which sets must be Loeb measurable?

Let Σ be the smallest o-algebra generated by all internal subsets of Ω. Then every set in Σ is Loeb measurable.

Let $\Sigma_0^1 = \prod_0^1$ =the set of all internal subsets of Ω^n for some $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ let

$$\Sigma_{n+1}^1 = \left\{ \left\{ x : \exists y(\langle x, y \rangle \in A) \right\} : A \in \prod_n^1 \right\}$$

and

$$\prod_{n+1}^1 = \left\{ \Omega \setminus A : A \in \Sigma_{n+1}^1 \right\}.$$

Fact:

Every set in $\sum_1^1$ and in $\prod_1^1$ is Loeb measurable.

All sets in $\sum$ are called Loeb-Bore1 sets and all sets in $\bigcup_{n=1}^{\infty} \sum_n^1$ are called projective sets.

Question:

Is it possible that every projective set is Loeb measurable?

Countably Determined Sets

A set A $\subset \Omega$ is called countably determined if there is an internal set $A_\sigma \subseteq \Omega$ for each $\sigma \in 2^{<\mathbb{N}}$ and there is a set $I \subseteq 2^{\mathbb{N}}$ such that

$$A = \bigcup_{f \in I} \bigcap_{n \in \mathbb{N}} A_{f|n}.$$

Fact:

(1) All projective subsets of Ω are countably determined.

(2) There are non-Loeb measurable countably determined sets.

Question:

Are there countably determined U-choice sets?

Known Results:

Theorem 1 (Keisler & Leth):

Let $U \subseteq \Omega$ be a cut. If $cof(U) \geq \aleph_1$, then there are no countably determined U-choice sets.

Steps of the proof:

(1) Suppose A is a U-choice set and A $= \bigcup_{f \in I} \bigcap_{n \in \mathbb{N}} A_{f|n}$.

(2) For each $f \in I$ there is an $n_f \in \mathbb{N}$ such that any two distinct elements in $\bigcap_{k \leq n} A_{f|k}$ have distance greater than U.

(3) There is a countable set $S \subseteq 2^{<\mathbb{N}}$ such that $\bigcup_{\sigma \in S} \bigcap_{k \leq |\sigma|} A_{\sigma|k}$ contains A and misses an entire U-monad.

Theorem 2 (Keisler & Leth) :

For any cut $U \subseteq \Omega$, there are no Σ_1^1 or Π_1^1 U-choice sets.

Steps of the proof:

(1) The case of $cof(U) \geq \aleph_1$ is covered in theorem 1. So one can assume $cof(U) = \aleph_0$.

(2) Let $\langle a_n : n \in \mathbb{N} \rangle$ be an increasing sequence cofinal in U. Then there is an *n* such that the set $D_{A,n} = \{ y : \exists x \in A \, (|y - x| < a_n) \}$ is Loeb measurable since it is Σ_1^1 or Π_1^1 and has positive Loeb measure by countable additivity.

(3) $D_{A.n}$ cannot be Loeb measurable by translation invariance of the Loeb measure.

Question (Kanovei):

Are there countably determined N-choice sets?

New Results:

Theorem 3

(1) If $U = x \cdot \mathbb{N}$ for some x, then there is a U-choice set.

(2) If $cof(U) = \aleph_0$ and $U \neq x \cdot \mathbb{N}$ for any x, then there are no U-choice sets.

Countably Determined

Steps of the proof:

(1) Case 1: $U = \mathbb{N}$. Let $I \subseteq 2^{\mathbb{N}}$ be a maximal set of pairwisely eventually different functions from $\mathbb{N}$ to 2. For each $\sigma \in 2$" let

$$A_\sigma = \left\{ 2^n K + \sum_{i<n} 2^i \sigma(i) : K \in {}^*\mathbb{N} \right\} \cap \Omega.$$

Then $A = \bigcup_{f \in I} \bigcap_{n \in \mathbb{N}} A_{f|n}$ is a $\mathbb{N}$-choice set.

Case 2: $U = x \cdot \mathbb{N}$. Then $B = x \cdot A$ is a U-choice set, where A is a $\mathbb{N}$-choice set.

(2)The proof is a variation of the proof of Theorem 1.

Question again:

What can be said about the Loeb measurability of projective sets?