Some Ramsey Theory in Nonstandard Analysis

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Outline of talk:

1. How does an 'applied' nonstandard analyst get sucked into the world of infinitary combinatorics?
2. Reflections on the use of NSA in infinitary combinatorics. (In particular: sketches of nonst'd proofs of results of Rowbottom et al.)

Motivation and History

- Common equation:

  Combinatorics+*NSA* = (*standard*) *mathematical results*

  — usually *finitary* combinatorics

- Some appearances of *infinitary* combinatorics within NSA:

  (I) Advantages to building the nonstandard model using special ultrafilters:

  - (Henson and Wattenberg, 1981) Measure theory in *selective* models.
  - (Benedikt 1993, 1995, 1998, 1999) Extensions of these results; other interesting ultrafilters. (c.f. Panetta 1978, Puritz 1971, et al.)
  - (Cutland, Kessler, Kopp, R. 1988) TFAE:
    1. $\forall x \in \mathbb{R}^{\mathbb{N}}/D$, if $x \approx 0$ then $x = {}^*s_M$ for some standard sequence s_n converging to 0 and some infinite M;
    2. D is a *P-point*

- (II) *Infinitary* combinatorics + NSA = (standard) mathematical results:

  - (R. 1983, 1996) Applications of Erdös-Hajnal to applied Loeb measure theory

  - (Benedikt) Recent work on database query languages (W/Keisler, Libkin, et al.) (Ramsey theory, Vapnik-Chervonenkis Dimension)

- (III) Use of NSA to prove Ramsey-like results:

  - (Keisler, Kunen, Miller, Leth 1989) Let Ω internal; then any $F : \mathcal{P}_n(\Omega) \to \mathbb{N}$ with a countably determined graph has an infinite internal homogeneous subset.

  - (Hirshfeld 1988) Nonstandard proofs of Ramsey's Theorem and related results.

Nonst'd proofs of results of Rowbottom et al.

Fix κ a cardinal number $\geq \omega$

Theorem. If κ is measurable then $(\kappa)^n_m$. In other words, if $F : \mathcal{P}_n(\kappa) \longrightarrow m$, then there is an $X \subseteq \kappa$ such that $\mathrm{card}(X) = \kappa$ and X is homogenous for F. $(n \in \mathbb{Z}^+, m < \kappa)$. Moreover, if μ is normal on κ then we may take $\mu(X) = 1$.

Definition. A measure μ on κ is *normal* provided that for any $f \in {}^\kappa\kappa$, if $\mu\{\alpha < \kappa \mid f(\alpha) < \alpha\} = 1$ then for some $\alpha_0 \in \kappa,\ \mu(f^{-1}\{\alpha_0\}) = 1$

Theorem. If κ is an uncountable measurable cardinal then κ is measurable with respect to a normal measure.

Corollary. (Erdös-Hajnal) Assume κ measurable, $\delta < \kappa$, $\forall \alpha < \delta, n_\alpha \in \mathbb{N}^+$, $m_\alpha < \kappa$, and $F_\alpha : \mathcal{P}_{n_\alpha}(\kappa) \longrightarrow m_\alpha$. Then there is an $X \subseteq \kappa$ such that $\mathrm{card}(X) = \kappa$ and X is homogenous for F_α (same X works for all α).

If $H \in {}^*\kappa$, define $\mu_H : \mathcal{P}(\kappa) \longrightarrow 2$ by:

$$\mu_H A = \begin{cases} 1, & H \in {}^*A \\ 0 & \text{otherwise,} \end{cases}$$

and say that H **represents** μ_H

Proposition. (Nonst'd representation of measures.) If $\mu : \mathcal{P}(\kappa) \longrightarrow 2$ is a finitely additive measure then for some $H \in {}^*\kappa,\ \mu = \mu_H$.

Proposition. (Nonst'd representation of normality.)The following are equivalent:

1. μ_H is normal
2. for any $f \in {}^\kappa\kappa$, if ${}^*f(H) < H$ then ${}^*f(H) \in \kappa$.

(In this case, call H normal.)

Proposition. (Nonst'd representation of α-additivity) The following are equivalent:

1. μ_H α–additive
2. $\forall \{A_i\}_{i<\alpha} \subseteq \mathcal{P}(\kappa),\ H \in (\bigcap_{i<\alpha} {}^*A_i) \Rightarrow H \in {}^*\left(\bigcap_{i<\alpha} A_i\right)$.

Corollary. If μ_H is α-additive and $f \in {}^\kappa\kappa$, then $\mu_{f(H)}$ is α-additive.

Theorem. If $H \in {}^*\kappa \setminus \kappa$ and μ_H is α–additive, $\alpha \geq \omega$, then for some normal $H_0 \in {}^*\kappa \setminus \kappa,\ \mu_{H_0}$ is α–additive.

Proof sketch:

Corollary. If κ is an uncountable measurable cardinal then κ is measurable with respect to a normal measure.

Proof of Rowbottom result:

(**Remark.** It suffices to assume m finite, $n = 2$.)

Let $\mu = \mu_H$ be a measure on κ. Put $C_j = \{x \in \kappa \mid {}^*F(x, H) = j\}$, $j < m$. Since $\kappa = C_0 \cup \cdots \cup C_{m-1}$, $H \in {}^*C_j$ for some j.

Define $\mathcal{U}_\alpha \subseteq \kappa$ inductively, $\alpha < \kappa$, so that:

(a) $H \in {}^*\mathcal{U}_\alpha$

(b) $\{x_\alpha\}_{\alpha<\kappa}$ is increasing, where $x_\alpha = \inf \mathcal{U}_\alpha$

(c) $F(x_\beta, x) = j$ whenever $x \in \mathcal{U}_\alpha$, $\alpha > \beta$

Here's how:

1. $\mathcal{U}_0 := C_j$.
2. α a limit: $\mathcal{U}_\alpha := \bigcap\limits_{\beta<\alpha} \mathcal{U}_\beta$.
3. $\alpha = \beta + 1$: $\mathcal{U}_\alpha := \{x \in \mathcal{U}_\beta \setminus \{x_\beta\} \mid F(x_\beta, x) = j\}$ (Note $x_\beta \in \mathcal{U}_\beta \subseteq C_j$, so ${}^*F(x_\beta, H) = j$, so $H \in {}^*\mathcal{U}_\alpha$.)

Put $X = \{x_\alpha\}_{\alpha<\kappa}$. (c) guarantees that $F(x, y) = j$ whenever $x \neq y \in X$, i.e. X is homogeneous for F.

Suppose now that μ is normal but $\mu X = 0$. Note that for any $x \in \mathcal{U}_0$ there is a greatest $\beta = \beta(x) < \kappa$ with $x \in \mathcal{U}_\beta$. (Otherwise, take α least with $x \notin \mathcal{U}_\alpha$, then α is a limit and $x \in \bigcap\limits_{\beta<\alpha} \mathcal{U}_\beta$, a contradiction.) Put $\varphi(x) = x_{\beta(x)}$, note $\varphi(x) < x$ for $x \in \mathcal{U}_0 \setminus X$, so ${}^*\varphi(H) = \alpha_0 = x_{\beta_0}$ for some α_0, $\beta_0 \in \kappa$; then $H \notin {}^*\mathcal{U}_{\beta_0+1}$, a contradiction.

Question: Suppose $f : (\Omega, \mathcal{A}_L, P_L) \longrightarrow Y$ is Loeb measurable, Y a metric space; does f have a lifting?

Suffices: (*) If $\mathcal{E}$ partitions Ω into Loeb nullsets, then for some $\mathcal{E}' \subseteq \mathcal{E}$, $\bigcup \mathcal{E}'$ is not Loeb measurable.

Remark: Follows easily if Loeb measure is *compact*; this can depend on underlying set theory (Jin, Shelah).

Theorem. (R., 1996) Suppose (i) $\kappa = \text{card}(\mathcal{A})$; (ii) For some nondecreasing sequence $\{\mathcal{A}_i\}_{i<\kappa}$ with each $\mathcal{A}_i \subseteq \mathcal{A}$ compact, $\mathcal{A} = \bigcup_{i<\kappa} \mathcal{A}_i$; and (iii) No $\alpha < \kappa$ is both measurable and cofinal in κ. Then (*) holds.

Proof start:

Else let $\mathcal{E} = \{E_i\}_{i<\alpha}$ be a counterexample with α least.

Induces a (σ-additive) measure on $(\alpha, \mathcal{P}(\alpha))$.

Ulam dichotomy:

(1) Atomless: RVMC, diagonalize using a Bernstein set.

(2) An atom: then WOLG α is a measurable cardinal, use Erdös-Hajnal in a clever way.

Question: Can (i)–(iii) in the theorem statement be replaced with (eg) the Isomorphism Property (Henson, Jin, Shelah) or some version of the Generic Filter Property (Di Nasso, Hrbacek)?