Nonstandard Analysis and Groups (mainly results of G. Keller)

(I) GROUPS

1. \( H \subseteq G \) generates \( G \) if \( G \) is the smallest subgroup of \( G \) which contains \( H \).

2. If \( e \in H = H^{-1} \), then \( H \) generates \( G \) provided \( G = \bigcup_n H^n \).

3. \( G \) is finitely generated provided there is a finite \( H \) which generates \( G \).

4. A word \( w(x_1, x_2, \ldots, x_n) \) is an identity relation (or law) for \( G \) provided \( \forall a_1, \ldots, a_n \in G, w(a_1, \ldots, a_n) = e \).

5. If \( L \) is a set of words, then \( V(L) \) = the variety for \( L \) = the class of all groups satisfying every law in \( L \).

6. If \( V \) is a variety of groups, \( F_n(V) \) is the reduced free group on \( n \) generators (i.e., the quotient of \( F_n \) by all the laws defining \( V \)).

7. The group \( G \) is amenable if there is a nontrivial left-invariant finitely-invariant measure on \( (G, \mathcal{P}(G)) \).

8. Theorem (Følner): \( G \) is amenable if and only if:

\[
\forall A \subseteq G \text{ finite } \forall r < 1 \exists E \subseteq G \text{ finite } \forall a \in A \frac{|E \cap aE|}{|E|} > r
\]

9. EG: \( \mathbb{Z}, SL(1, \mathbb{R}), SL(2, \mathbb{R}) \) are amenable; \( F_2 \) is not amenable; a group \( G \subseteq SL(n, \mathbb{R}) \) of isometries of \( \mathbb{R}^n \) is amenable if and only if \( F_2 \not\subseteq G \); homomorphic images and subgroups of amenable groups are amenable.

10. Call a group \( G \) uniformly Følner, or uniformly amenable (UA) if \( |E| \) can be chosen to depend only on \( |A| \) and \( r \), that is, if there is a function \( F : \mathbb{N} \times (0,1) \rightarrow \mathbb{N} \) such that

\[
\forall n \in \mathbb{N} \forall A \subseteq G \text{ s.t. } |A| < n \forall r < 1 \exists E \subseteq G \text{ s.t. } |E| < F(n, r) \text{ and } \forall a \in A \frac{|E \cap aE|}{|E|} > r
\]
11. A class $\mathcal{D}$ of groups is uniformly amenable if there is a single function $F : \mathbb{N} \times (0,1) \to \mathbb{N}$ that witnesses UA for all the groups in $\mathcal{D}$.

(II) Nonstandard Analysis

Start with a mathematical universe (superstructure) $V$, containing:

- All natural numbers $0, 1, 2, \ldots$; real numbers $\sqrt{2}, \pi, e, \phi, \ldots$; etc.
- The set $\mathbb{N}$ of natural numbers as an object; the set $\mathbb{R}$ of real numbers; etc.
- Every function from $\mathbb{R}$ to $\mathbb{R}$, and the set of all such functions
- Your favorite groups, Banach spaces, etc
- Every other mathematical object we might want to talk about
- Closure under $\epsilon, \mathcal{P}$, etc.

We call the elements of this mathematical universe standard.

Extend to a nonstandard mathematical universe $^*V$:

- For every object $A$ in $V$, there is a corresponding object $^*A$ in $^*V$
- EG, $^*V$ has objects $^*\mathbb{N}$, $^*\mathbb{R}$, $^*\sin(x)$, etc.
- (For simplicity, we drop the stars from simple objects like numbers: $12$ instead of $^*12$ etc)
- There may (generally will) be many more objects in $^*V$ than in $V$
- An element of $^*V$ that is not in $V$ is called nonstandard.

The extension should satisfy two important properties:

Transfer If $S$ is a bounded first-order statement about objects in $V$, then $S$ is true in $V$ if and only if it true in $^*V$.

For example, let $(G, \cdot, e)$ be a multiplicative group; the following are true in $V$: 
\((\forall x \in G)(\exists y \in G)[(x \cdot y = e) \land (y \cdot x = e)]\)
\((\forall x \in G)[(x \cdot e = x) \land (x \cdot e = x)]\)
\((\forall x \in G)(\forall y \in G)(\forall z \in G)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]\)

By transfer it follows:
\((\forall x \in ^{*}G)(\exists y \in ^{*}G)[(x^* \cdot y = ^{*}e) \land (y^* \cdot x = ^{*}e)]\)
\((\forall x \in ^{*}G)[(x^* \cdot e = x) \land (x^* \cdot e = x)]\)
\((\forall x \in ^{*}G)(\forall y \in ^{*}G)(\forall z \in ^{*}G)[(x^* \cdot y)^* \cdot z = x^* \cdot (y^* \cdot z)]\)

In other words, \(^{*}G\) is also not only a \(^{*}\)group, but also an actual group.

As another example, since 12 is an element of \(\mathbb{N}\), \(^{*}12\) is an element of \(^{*}\mathbb{N}\).

Since we can think of the basic elements (like \(^{*}12\)) of \(^{*}\mathbb{V}\) as just being
the same as their counterparts (like 12) in \(\mathbb{V}\), \(^{*}\mathbb{N}\) is a superset of \(\mathbb{N}\).

Similarly, for any standard set \(A\) which is an object of \(\mathbb{V}\), the set \(^{*}A\) in
\(^{*}\mathbb{V}\) extends the set \(A\).

**Saturation:**

A set \(a \subseteq ^{*}\mathbb{V}\) is internal if \(\exists b \in \mathbb{V} \ a \in ^{*} b\) (otherwise it is external)

For example, if \(A \in \mathbb{V}\) then \(\mathcal{P}(A) \in \mathbb{V}\), so \(^{*}A \in ^{*}\mathcal{P}(A)\) holds, and \(^{*}A\) is internal.

Equivalently, a set \(a\) is internal if it can be defined from other internal
sets by a bounded first-order formula.

Now, \(\kappa\)-saturation is the property:

If \(\mathcal{A}\) is a set of sets with the finite intersection property, and \(|\mathcal{A}| < \kappa\),
then \(\bigcap \mathcal{A} \neq \emptyset\).

Equivalently, any set of statements of cardinality < \(\kappa\) about an object \(X\)
which is finitely satisfiable in \(^{*}\mathbb{V}\), can all be simultaneously satisfied
by a single object in \(^{*}\mathbb{V}\).
We will always assume that the model is \( \kappa \)-saturated for \( \kappa \) bigger than the cardinality of every standard set (though much less saturation usually suffices).

Saturation roughly means: Anything that can happen in \( \ast V \), does happen.

**Example:** Consider the statements:

- \( x \) is a real number
- \( x > 0 \)
- \( x < 1 \)
- \( x < 1/2 \)
- \( x < 1/3 \)
- \( x < 1/4 \)
  
Any finite set of these statements refers to a smallest fraction \( 1/N \); but
  
then, \( x = \frac{1}{N+1} \) satisfies this finite set of statements.

It follows that there is a an element of \( \ast \mathbb{R} \), call it \( \epsilon \), such that

\[ \epsilon > 0 \]

and, for every (standard) natural number \( N \),

\[ \epsilon < 1/N \]

We have proved that \( \ast \mathbb{R} \) contains nonzero infinitesimals, where

**Definition:** An infinitesimal is an element \( \epsilon \) of \( \ast \mathbb{R} \) such that

\[ |\epsilon| < 1/N \]

for every natural number \( N \) in \( \mathbb{N} \)

Since \( \ast \mathbb{R} \) (sometimes called the set of “hyperreal numbers”) is, like the usual set of real numbers, closed under the basic arithmetic operations, it also contains negative infinitesimals (like \( -\epsilon \)), infinite numbers (like \( 1/\epsilon \)), and many other objects:
In particular, as we have seen there are elements of $\mathbb{N}^*$ which are bigger than every element of $\mathbb{N}$; in other words, there are infinite integers.

Many applications are based on the ubiquity of “hyperfinite sets”

**Definition:** A set $E$ in $\mathbb{V}^*$ is hyperfinite if there is a *one-to-one correspondence* between $E$ and $\{0, 1, 2, \ldots, H\}$ for some $H$ in $\mathbb{N}^*$. Equivalently, if the mathematical statement “$E$ is finite” holds in $\mathbb{V}$.

**Examples:**
1. Every finite set is hyperfinite.
2. If $H$ is an infinite integer, $\{0, 1, 2, \ldots, H\} = \{n \in \mathbb{N}^* : n \leq H\}$ is a hyperfinite subset of $\mathbb{N}^*$.
3. If $H$ is an infinite integer, $\{0, \frac{1}{H}, \frac{2}{H}, \ldots, \frac{H-1}{H}, 1\}$ is a hyperfinite subset of $\mathbb{V}^*[0,1]$.

**Theorem:** If $A$ is an infinite set in $\mathbb{V}$, then there is a hyperfinite set $\hat{A}$ in $\mathbb{V}^*$ such that every element of $A$ is in $\hat{A}$.

**Proof:** Consider the statements: (i) $X$ is finite; (ii) $a \in X$ (one such statement for every element $a$ of $A$).

Given any finite number of these statements, a corresponding finite number $\{a_1, \ldots, a_n\}$ of elements of $A$ are mentioned, so $X = \{a_1, \ldots, a_n\}$ satisfies those statements. By the saturation principle, there is therefore a set $X$ in $\mathbb{V}$ satisfying all the statements simultaneously; let $\hat{A}$ be this $X$. ◼

**Corollary:** There is a hyperfinite set containing $\mathbb{R}$.

“Nonstandard analysis is the art of making infinite sets finite by extending them.” —M. Richter
Goal: **Theorem:** let $V$ be a variety of groups. Then $V$ is UA iff $V$ is amenable.

Nonstandard motivation:

Let $G$ be a group, and suppose the group $^*G$ is (externally) amenable. That is, there is a measure $\mu : \mathcal{P}(^*G) \to \mathbb{R}$ such that

$$(\forall g \in ^*G)[\mu(E) = \mu(aE)]$$

Then $\nu(A) := \mu(^*A)$ is evidently a left-invariant measure on $G$. This proves:

**Proposition:** If $^*G$ is amenable then $G$ is amenable.

Question: Does $G$ amenable imply $^*G$ is amenable?

Answer: No. Example later.

**Theorem:** Let $G$ be a group. TFAE: (1) $G$ is UA; (2) $^*G$ is UA; (3) $^*G$ is amenable.

**Proof.** (1 $\Rightarrow$ 2) Let $F$ witness UA of $G$. Claim: $F$ witnesses UA of $^*G$ as well. Let $n, r$ be given, and let $A \subseteq ^*G$ with $|A| < n$. By transfer, $^*F : \mathbb{N} \times (0, 1) \to ^*\mathbb{N}$ witnesses $^*UA$, so

$$\exists E \in ^*\mathcal{P}(G), |E| \leq ^*F(n, r) \ \& \ \forall a \in A \ \frac{|E \cap aE|}{|E|} > r.$$ 

Note that an internal subset $E$ of $^*G$ which has internal cardinality $\leq ^*F(n, r)$ is externally finite with an actual, standard finite cardinality less than $F(n, r)$, since $n$ and $r$ are standard and $^*F(n, r) = F(n, r)$. This proves the claim.

(2 $\Rightarrow$ 3) is trivial.

(3 $\Rightarrow$ 1) Let $n \in \mathbb{N}, r < 1$ be given. We need to define $F(n, r)$. Let $m \in ^*\mathbb{N} \setminus \mathbb{N}$. By amenability of $^*G$ and the Følner condition,

$$\forall A \in \mathcal{P}(G) |A| < n \Rightarrow \exists E \in \mathcal{P}(G), |E| \text{ finite} \ & \ \forall a \in A \ \frac{|E \cap aE|}{|E|} > r.$$
Since any subset of $^*G$ with (standard) finite cardinality is internal, and any finite set has cardinality less than $m$, it follows that

\[ \exists m \in ^* \mathbb{N} \forall A \in ^* \mathcal{P}(G) |A| < n \Rightarrow \exists E \in ^* \mathcal{P}(G), |E| \text{ finite } \& \forall a \in A \frac{|E \cap aE|}{|E|} > r. \]

By transfer, there is a standard finite $m$ that works for this $n$ and $r$; put $F(n, r) := m$.

**Corollary** A subgroup or homeomorphic image of a UA group is UA.

**Proof.** If $H$ is a subgroup of $G$, and $G$ is UA, then $^*G$ is amenable, $^*H$ must (as an external group) be amenable, so $H$ is UA. A similar argument works for homeomorphic images (since the homeomorphic image of an amenable group is amenable).

**Theorem** Let $\mathcal{G}$ be a set of groups; then $\mathcal{G}$ is uniformly amenable iff $^*\mathcal{G}$ is amenable.

**Proof.** ($\Rightarrow$) If $F$ witnesses UA for $\mathcal{G}$ then it witnesses amenability for every $G \in ^* \mathcal{G}$ as in the proof of the last theorem.

($\Leftarrow$) Fix $n \in \mathbb{N}, r < 1$ be given. We need to define $F(n, r)$ and $m \in ^* \mathbb{N} \setminus \mathbb{N}$. By amenability of $^*\mathcal{G}$ and the Følner condition, $m$ witnesses

\[ \exists m \in ^* \mathbb{N} \forall G \in ^* \mathcal{G} \forall A \in ^* \mathcal{P}(G) |A| < n \Rightarrow \exists E \in ^* \mathcal{P}(G), |E| \text{ finite } \& \forall a \in A \frac{|E \cap aE|}{|E|} > r \]

as above. By transfer, there is a standard finite $m$ that works for this $n$ and $r$; put $F(n, r) := m$.

**Proposition** If $\mathcal{V}$ is a variety of groups and $\mathcal{G} \subseteq \mathcal{V}$ then $^*\mathcal{G} \subseteq \mathcal{V}$

**Proof.** Let $\ell : w(x_1, \ldots, x_n)$ be a law for $\mathcal{V}$, that is, $\ell \in \mathcal{L}$ where $\mathcal{V} = \mathcal{V}(\mathcal{L})$. Now,

\[ \forall G \in \mathcal{G} \forall g_1, \ldots, g_n \in G \ [w(g_1, \ldots, g_n) = e] \]

so by transfer,

\[ \forall G \in ^* \mathcal{G} \forall g_1, \ldots, g_n \in G \ [w(g_1, \ldots, g_n) = e] \]

so every $G \in ^* \mathcal{G}$ satisfies $\ell$. Since $\ell$ was arbitrary in $\mathcal{L}$, $^*\mathcal{G} \subseteq \mathcal{V}$. 

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**Corollary** Let $V$ be a variety of groups. Then $V$ is UA iff $V$ is amenable.

**Proof.** $(\Leftarrow)$ is trivial. $(\Rightarrow)$ Let $\mathcal{F} = N \times (0, 1)^N$ be the set of all functions from $N \times (0, 1)$ to $N$. Suppose $V$ is not UA, then for every $F \in \mathcal{F}$ there is a $G_F \in V$ such that $F$ does not witness UA for $G_F$. Let $\mathcal{G} = \{G_F\}_{F \in \mathcal{F}}$. Clearly $\mathcal{G}$ is not UA, so $^{*}\mathcal{G}$ is not amenable. But $^{*}\mathcal{G} \subseteq V$ by the last proposition, so $V$ is not amenable.
An example

Let $G = \{ \pi \in \text{Permutations}(\mathbb{N}) : \exists N \in \mathbb{N} \forall x > N \ x(\pi) = x \}$. 

Claim 1: $G$ is amenable. One way to see this is to note that every finitely-generated subgroup of $G$ is finite, so trivially amenable, and by standard nonsense this implies that $G$ is amenable. Or, use Følner: Let $A = \{ a_1, \ldots, a_n \} \subseteq G, r < 1$. For some sufficiently large $N$ and all $x < N$ and $i \leq n$, $a_i(x) = x$. let $E = \{ 0, \ldots, M \}$, where $M > (N + r + 1)/(1 - r)$. Now, if $a \in A$ then $E \cap aE \supseteq \{ N + 1, \ldots, M \}$, so $\frac{|E \cap aE|}{|E|} \geq (M - N - 1)M + 1 > r$ by the choice of $M$.

Claim 2: $\ast G$ is not amenable. It suffices to find $F_2 \subseteq \ast G$, where $F_2$ is freely generated by $\{ a, b \}$. Let $M \in^* \mathbb{N} \setminus \mathbb{N}$, let $\hat{F}$ be the (internal) set of all words of length at most $M$ from $\{ a, b, a^{-1}, b^{-1} \}$. Write $\hat{F} = \{ f_0, \ldots, f_{H-1} \}$ (where $H$ is the internal cardinality of $\hat{F}$, and $f_0 = e$), and identify this set with $\{ 0, \ldots, H - 1 \}$. Let $\hat{F}_a = \{ g \in \hat{F} | ag \in \hat{F} \}$, and $\hat{F}_b = \{ g \in \hat{F} | bg \in \hat{F} \}$. There is an internal bijection $\hat{a} : \hat{F} \to \hat{F}$ such that $\hat{a}(g) = ag$ for every $g \in \hat{F}_a$. Same for $\hat{b}$. Note $F_2 \subseteq \hat{F}_a \cap \hat{F}_b$. By the identification above, $\hat{a}, \hat{b} \in^* G$. Claim: if $w(x, y)$ is a word and $f_i \in F_2$, then $w(a, b)f_i = f_{w(\hat{a}, \hat{b})(i)}$. The proof is an easy induction on the length of $w$. It follows that if $w(\hat{a}, \hat{b}) = id$ then $w(a, b) = e$, and this proves that $\hat{a}$ and $\hat{b}$ generate a free group.

Thus, $G$ is an example of a group which is amenable but not UA.