Nonstandard Analysis and Groups (mainly results of G. Keller)

(I) GROUPS

1. $H \subseteq G$ *generates* G if G is the smallest subgroup of G which contains H.

2. If $e \in H = H^{-1}$, then H generates G provided $G = \bigcup_n H^n$

3. G is *finitely generated* provided there is a finite H which generates G.

4. A word $w(x_1, x_2, \ldots, x_n)$ is an *identity relation* (or *law*) for G provided $\forall a_1, \ldots, a_n \in G, w(a_1, \ldots, a_n) = e$.

5. If L is a set of words, then $V(L)$ =the variety for L=the class of all groups satisfying every law in L.

6. If V is a variety of groups, $F_n(V)$ is the *reduced free group* on n generators (ie, the quotient of F_n by all the laws defining V)

7. The group G is *amenable* if there is a nontrivial left-invariant finitely-invariant measure on $(G, \mathcal{P}(G))$

8. Theorem (Følner): G is amenable if and only if:

$$\forall A \subseteq G \text{ finite } \forall r < 1 \exists E \subseteq G \text{ finite } \forall a \in A \ \frac{|E \cap aE|}{|E|} > r$$

9. EG: $\mathbb{Z}, SL(1, \mathbb{R}), SL(2, \mathbb{R})$ are amenable; F_2 is not amenable; a group $G \subseteq SL(n, \mathbb{R})$ of isometries of $\mathbb{R}^n$ is amenable if and only if $F_2 \not\subseteq G$; homomorphic images and subgroups of amenable groups are amenable.

10. Call a group G *uniformly Følner*, or *uniformly amenable* (UA) if $|E|$ can be chosen to depend only on $|A|$ and r, that is, if there is a function $F : \mathbb{N} \times (0, 1) \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \ \forall A \subseteq G \text{ s.t. } |A| < n \ \forall r < 1$$

$$\exists E \subseteq G \text{ s.t. } |E| < F(n, r) \ \& \ \forall a \in A \ \frac{|E \cap aE|}{|E|} > r$$

11. A class $\mathcal{D}$ of groups is *uniformly amenable* if there is a single function $F : \mathbb{N} \times (0,1) \to \mathbb{N}$ that witnesses UA for all the groups in $\mathcal{D}$

(II) Nonstandard Analysis

Start with a mathematical universe (*superstructure*) V, containing:

- All natural numbers $0, 1, 2, \ldots$; real numbers $\sqrt{2}, \pi, e, \phi, \ldots$; etc.
- The *set* $\mathbb{N}$ of natural numbers *as an object*; the set $\mathbb{R}$ of real numbers; etc.
- Every function from $\mathbb{R}$ to $\mathbb{R}$, and the set of all such functions
- Your favorite groups, Banach spaces, etc
- Every other mathematical object we might want to talk about
- Closure under $\in, \mathcal{P}$, etc.
- We call the elements of this mathematical universe *standard*.

Extend to a *nonstandard* mathematical universe $^*V$:

- For every object A in V, there is a corresponding object $^*A$ in $^*V$
- EG, $^*V$ has objects $^*\mathbb{N}$, $^*\mathbb{R}$, $^*\sin(x)$, etc.
- (For simplicity, we drop the stars from simple objects like numbers: 12 instead of ^{*12} etc)
- There may (generally will) be many more objects in $^*V$ than in V
- An element of $^*V$ that is **not** in V is called *nonstandard*.

The extension should satisfy two important properties:

Transfer If S is a bounded first-order statement about objects in V, then S is true in V if and only if it true in $^*V$

For example, let $(G, \cdot, e)$ be a multiplicative group; the following are true in V:

$(\forall x \in G)(\exists y \in G)[(x \cdot y = e) \wedge (y \cdot x = e)]$

$(\forall x \in G)[(x \cdot e = x) \wedge (x \cdot e = x)]$

$(\forall x \in G)(\forall y \in G)(\forall z \in G)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$

By transfer it follows:

$(\forall x \in^* G)(\exists y \in^* G)[(x {}^*\!\cdot y =^* e) \wedge (y {}^*\!\cdot x =^* e)]$

$(\forall x \in^* G)[(x {}^*\!\cdot {}^*e = x) \wedge (x {}^*\!\cdot {}^*e) = x)]$

$(\forall x \in^* G)(\forall y \in^* G)(\forall z \in^* G)[(x {}^*\!\cdot y) {}^*\!\cdot z = x {}^*\!\cdot (y {}^*\!\cdot z)]$

In other words, $^*G$ is also not only a *group, but also an actual group.

As another example, since 12 is an element of $\mathbb{N}$, ^{*12} is an element of $^*\mathbb{N}$.

Since we can think of the basic elements (like ^{*12}) of $^*V$ as just being the same as their counterparts (like 12) in V, $^*\mathbb{N}$ is a superset of $\mathbb{N}$.

Similarly, for *any* standard set A which is an object of V, the set $^*A$ in $^*V$ extends the set A.

Saturation:

A set $a \subseteq^* V$ is *internal* if $\exists b \in V \;\; a \in^* b$ (otherwise it is *external*)

For example, if $A \in V$ then $\mathcal{P}(A) \in V$, so $^*A \in^* \mathcal{P}(A)$ holds, and $^*A$ is internal.

Equivalently, a set a is internal if it can be defined from other internal sets by a bounded first-order formula.

Now, κ-saturation is the property:

If $\mathcal{A}$ is a set of sets with the finite intersection property, and $|\mathcal{A}| < \kappa$, then $\bigcap \mathcal{A} \neq \emptyset$.

Equivalently, any set of statements of cardinality $< \kappa$ about an object X which is finitely satisfiable in $^*V$, can all be simultaneously satisfied by a single object in $^*V$

We will always assume that the model is κ—saturated for κ bigger than the cardinality of every standard set (though much less saturation usually suffices).

Saturation roughly means: Anything that can happen in *V, *does* happen.

Example: Consider the statements:

$$x \text{ is a real number}$$

$$x > 0$$

$$x < 1$$

$$x < 1/2$$

$$x < 1/3$$

$$x < 1/4$$

$$\vdots$$

Any finite set of these statements refers to a smallest fraction $1/N$; but then, $x = \frac{1}{N+1}$ satisfies this finite set of statements.

It follows that there is a an element of $^*\mathbb{R}$, call it ϵ, such that

$$\epsilon > 0$$

and, for every (standard) natural number N,

$$\epsilon < 1/N$$

We have proved that $^*\mathbb{R}$ contains nonzero infinitesimals, where

Definition: An *infinitesimal* is an element ϵ of $^*\mathbb{R}$ such that

$$|\epsilon| < 1/N$$

for every natural number N in $\mathbb{N}$

Since $^*\mathbb{R}$ (sometimes called the set of *"hyperreal numbers"*) is, like the usual set of real numbers, closed under the basic arithmetic operations, it also contains negative infinitesimals (like $-\epsilon$), infinite numbers (like $1/\epsilon$), and many other objects:

In particular, as we have seen there are elements of $^*\mathbb{N}$ which are bigger than every element of $\mathbb{N}$; in other words, there are *infinite integers.*

Many applications are based on the ubiquity of "hyperfinite sets"

Definition: A set E in $^*V$ is *hyperfinite* if there is a *one-to-one correspondence between E and $\{0, 1, 2, \ldots, H\}$ for some H in $^*\mathbb{N}$. Equivalently, if the mathematical statement "E is finite" holds in $^*V$.

Examples: 1. Every finite set is hyperfinite.

2. If H is an infinite integer, $\{0, 1, 2, \cdots, H\} = \{n \in {}^*\mathbb{N} : n \leq H\}$ is a hyperfinite subset of $^*\mathbb{N}$

3. If H is an infinite integer, $\{0, \frac{1}{H}, \frac{2}{H}, \cdots, \frac{H-1}{H}, 1\}$is a hyperfinite subset of $^*[0.1]$

Theorem: If A is an infinite set in V then there is a hyperfinite set $\hat{A}$ in $^*V$ such that every element of A is in $\hat{A}$

Proof: Consider the statements: (i) X is finite; (ii) $a \in X$ (one such statement for every element a of A)

Given any finite number of these statements, a corresponding finite number $\{a_1, \ldots, a_n\}$ of elements of A are mentioned, so $X = \{a_1, \ldots, a_n\}$ satisfies those statements. By the saturation principle there is therefore a set X in $^*V$ satisfying all the statements simultaneously; let $\hat{A}$ be this X. $\dashv$

Corollary: There is a hyperfinite set containing $\mathbb{R}$.

> "Nonstandard analysis is the art of making infinite sets finite by extending them." —M. Richter

(III) BACK TO GROUPS

Goal: **Theorem:** let V be a variety of groups. Then V is UA iff V is amenable.

Nonstandard motivation:

Let G be a group, and suppose the group $^*G$ is (externally) amenable. That is, there is a measure $\mu : \mathcal{P}(^*G) \longrightarrow \mathbb{R}$ such that

$$(\forall g \in^* G)[\mu(E) = \mu(aE)]$$

Then $\nu(A) := \mu(^*A)$ is evidently a left-invariant measure on G. This proves:

Proposition: If $^*G$ is amenable then G is amenable.

Question: Does G amenable imply $^*G$ is amenable?

Answer: No. Example later.

Theorem: Let G be a group. TFAE: (1) G is UA; (2) $^*G$ is UA; (3) $^*G$ is amenable.

Proof. $(1 \Rightarrow 2)$ Let F witness UA of G. Claim: F witnesses UA of $^*G$ as well. Let n, r be given, and let $A \subseteq^* G$ with $|A| < n$. By transfer, $^*F :^* \mathbb{N} \times^* (0,1) \longrightarrow^* \mathbb{N}$ witnesses *UA, so

$$\exists E \in^* \mathcal{P}(G),\ \ |E| \leq^* F(n,r)\ \ \&\ \forall a \in A\ \ \frac{|E \cap aE|}{|E|} > r.$$

Note that an internal subset E of $^*G$ which has internal cardinality $\leq^* F(n,r)$ is externally finite with an actual, standard finite cardinality less than $F(n,r)$, since n and r are standard and $^*F(n,r) = F(n,r)$. This proves the claim.

$(2 \Rightarrow 3)$ is trivial.

$(3 \Rightarrow 1)$ Let $n \in \mathbb{N}, r < 1$ be given. We need to define $F(n,r)$. Let $m \in^* \mathbb{N} \setminus \mathbb{N}$. By amenability of $^*G$ and the Følner condition,

$$\forall A \in \mathcal{P}(G) |A| < n \Rightarrow \exists E \in \mathcal{P}(G),\ \ |E|\ \text{finite}\ \&\ \forall a \in A\ \ \frac{|E \cap aE|}{|E|} > r.$$

Since any subset of $^*G$ with (standard) finite cardinality is internal, and any finite set has cardinality less than m, it follows that

$$\exists m \in^* \mathbb{N} \forall A \in^* \mathcal{P}(G)|A| < n \Rightarrow \exists E \in^* \mathcal{P}(G),\ |E| \text{ finite } \&\ \forall a \in A\ \frac{|E \cap aE|}{|E|} > r.$$

By transfer, there is a standard finite m that works for this n and r; put $F(n, r) := m$.

Corollary A subgroup or homeomorphic image of a UA group is UA.

Proof. If H is a subgroup of G, and G is UA, then $^*G$ is amenable, $^*H$ must (as an external group) be amenable, so H is UA. A similar argument works for homeomorphic images (since the homeomorphic image of an amenable group is amenable).

Theorem Let $\mathcal{G}$ be a set of groups; then $\mathcal{G}$ is uniformly amenable iff $^*\mathcal{G}$ is amenable.

Proof. ($\Rightarrow$) If F witnesses UA for $\mathcal{G}$ then it witnesses amenability for every $G \in^* \mathcal{G}$ as in the proof of the last theorem.

($\Leftarrow$) Fix $n \in \mathbb{N}, r < 1$ be given. We need to define $F(n, r)$. and $m \in^* \mathbb{N} \setminus \mathbb{N}$. By amenability of $^*\mathcal{G}$ and the Følner condition, m witnesses

$$\exists m \in^* \mathbb{N}\ \forall G \in^* \mathcal{G}\ \ \forall A \in^* \mathcal{P}(G)|A| < n \Rightarrow \exists E \in^* \mathcal{P}(G),\ |E| \text{ finite } \&\ \forall a \in A\ \frac{|E \cap aE|}{|E|} > r$$

as above. By transfer, there is a standard finite m that works for this n and r; put $F(n, r) := m$.

Proposition If V is a variety of groups and $\mathcal{G} \subseteq V$ then $^*\mathcal{G} \subseteq V$

Proof Let $\ell : w(x_1, \ldots, x_n)$ be a law for V, that is, $\ell \in L$ where $V = V(L)$. Now,

$$\forall G \in \mathcal{G}\quad \forall g_1, \ldots, g_n \in G\ [w(g_1, \ldots, g_n) = e]$$

so by transfer,

$$\forall G \in^* \mathcal{G}\quad \forall g_1, \ldots, g_n \in G\ [w(g_1, \ldots, g_n) = e]$$

so every $G \in^* \mathcal{G}$ satisfies ℓ. Since ℓ was arbitrary in L, $^*\mathcal{G} \subseteq V$.

Corollary Let V be a variety of groups. Then V is UA iff V is amenable.

Proof. ($\Leftarrow$) is trivial. ($\Rightarrow$) Let $\mathcal{F} = {}^{\mathbb{N}\times(0,1)}\mathbb{N}$ be the set of all functions from $\mathbb{N}\times(0,1)$ to $\mathbb{N}$. Suppose V is *not* UA, then for every $F \in \mathcal{F}$ there is a $G_F \in V$ such that F *does not* witness *UA* for G_F. Let $\mathcal{G} = \{G_F\}_{F\in\mathcal{F}}$. Clearly $\mathcal{G}$ is not UA, so $^*\mathcal{G}$ is not amenable. But $^*\mathcal{G} \subseteq V$ by the last proposition, so V is not amenable.

An example

Let $G = \{\pi \in \text{Permutations}(\mathbb{N}) : \exists N \in \mathbb{N}\ \forall x > N\ \ \pi(x) = x\}$.

Claim 1: G is amenable. One way to see this is to note that every finitely-generated subgroup of G is finite, so trivially amenable, and by standard nonsense this implies that G is amenable. Or, use Følner: Let $A = \{a_1, \ldots, a_n\} \subset G, r < 1$. For some sufficiently large N and all $x < N$ and $i \leq n$, $a_i(x) = x$. let $E = \{0, \ldots, M\}$, where $M > (N + r + 1)/(1 - r)$. Now, if $a \in A$ then $E \cap aE \supseteq \{N + 1, \ldots, M\}$, so $\frac{|E \cap aE|}{|E|} \geq (M - N - 1)M + 1 > r$ by the choice of M.

Claim 2: ${}^*G$ is not amenable. It suffices to find $F_2 \subseteq^* G$, where F_2 is freely generated by $\{a, b\}$. Let $M \in^* \mathbb{N} \setminus \mathbb{N}$, let $\hat{F}$ be the (internal) set of all words of length at most M from $\{a, b, a^{-1}, b^{-1}\}$. Write $\hat{F} = \{f_0, \ldots, f_{H-1}\}$ (where H is the internal cardinality of $\hat{F}$, and $f_0 = e$), and identify this set with $\{0, \ldots, H - 1\}$.

Let $\hat{F}_a = \{g \in \hat{F} | ag \in \hat{F}\}$, and $\hat{F}_b = \{g \in \hat{F} | bg \in \hat{F}\}$. There is an internal bijection $\hat{a} : \hat{F} \to \hat{F}$ such that $\hat{a}(g) = ag$ for every $g \in \hat{F}_a$. Same for $\hat{b}$. Note $F_2 \subseteq \hat{F}_a \cap \hat{F}_b$. By the identification above, $\hat{a}, \hat{b} \in^* G$. Claim: if $w(x, y)$ is a word and $f_i \in F_2$, then $w(a, b)f_i = f_{w(\hat{a}, \hat{b})(i)}$. The proof is an easy induction on the length of w. It follows that if $w(\hat{a}, \hat{b}) = id$ then $w(a, b) = e$, and this proves that $\hat{a}$ and $\hat{b}$ generate a free group.

Thus, G is an example of a group which is amenable but not UA.