We now have:

Finite sets: $\text{card}(A) = \text{card}(\{0, 1, 2, \ldots, n-1\})$ for some $n \in \mathbb{N}$

Infinites sets: $\mathbb{N}, \mathbb{N} - \{0\}, \mathbb{Z}$ - all have the same cardinality

More infinite sets: $(0, 1), \mathbb{R}$, any line segment in plane - all have same cardinality

Questions: 1. Do $\mathbb{N}$ and $\mathbb{R}$ have the same cardinality? (Answer: No)

2. If not - what is the relationship between these *orders of infinity*?

3. **(Continuum hypothesis)** Is there a set E such that $\mathbb{N} \subset E \subset \mathbb{R}$ and $\text{card}(\mathbb{N}) \neq \text{card}(E) \neq \text{card}(\mathbb{R})$?

Theorem (Cantor): $\text{card}(\mathbb{R}) \neq \text{card}(\mathbb{N})$

Proof:

Definition: An infinite set A is *countable* (or *denumerable*, or *enumerable*) if it has the same cardinality as $\mathbb{N}$.

So: $\mathbb{N}, \mathbb{N} - \{0\}, \mathbb{Z}$ are countable.

$\mathbb{R}$ is *not* countable.

What about $\mathbb{Q}$?

Some useful (obvious?) facts

Theorem For any sets A, B, and C:

1. A has the same cardinality as A
2. If A has the same cardinality as B then B has the same cardinality as A
3. If A has the same cardinality as B and B has the same cardinality as C then A has the same cardinality as C

Definition: *A has no greater cardinality than B* (or $\text{card}(A) \leq \text{card}(B)$) provided A has the same cardinality as some subset of B, that is, for some $C \subseteq B, \quad \text{card}(A) = \text{card}(C)$