

## Some terminology

The *cardinal number* of a set  $A$  is the collection of all sets with the same cardinality as  $A$ . (Warning!)

A set is *countable* if it is either finite or countably infinite, otherwise it is *uncountable*.

A countably infinite set is said to have cardinal number (or *cardinality*)  $\aleph_0$

A set with same cardinality as  $\mathbb{R}$  is said to have *cardinality of the continuum* (or cardinal number  $\mathfrak{c}$ , or sometimes  $2^{\aleph_0}$ , for reasons that will be clear later).

**Two statements that show that  $\aleph_0$  is the smallest infinite cardinality**

- I.** Every infinite set contains a countably infinite subset.
- II.** Every subset of a countably infinite set is countable.

(Proof of I)

**(Cantor-Schröder-Bernstein)** If  $\text{card}(A) \leq \text{card}(B)$  and  $\text{card}(B) \leq \text{card}(A)$  then  $\text{card}(A) = \text{card}(B)$

**Examples:**

$(0, 1) \subseteq [0, 1] \subset \mathbb{R}$ , so...

Property II from previous page.

## Remark on Axiom of Choice

We know that for every infinite set  $A$ ,  $\text{card}(A) < \text{card}(\mathbb{N})$  or  $\text{card}(A) = \text{card}(\mathbb{N})$  or  $\text{card}(\mathbb{N}) < \text{card}(A)$ . (Why?)

In other words,  $\mathbb{N}$  is *comparable* to all other sets.

We'd like *all* sets to be comparable, since then cardinalities would 'line up' (picture)

**Zermelo's Axiom of Choice (AC):** If  $\mathcal{A}$  is any collection of disjoint nonempty sets, then there is a new set containing exactly one element from each of the given sets. (picture)

**Theorem:** If AC is true then all sets are comparable in the sense that  $\forall A \forall B (\text{card}(A) \leq \text{card}(B) \vee \text{card}(B) \leq \text{card}(A))$

**Theorem (Paul Cohen)** AC is independent of the other axioms of set theory.

*To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed.*

– Bertrand Russell