11.2.1 Parenthetical remark

Writing all the parentheses needed for some formulas can be very tedious, and reading a formula with all the parentheses correct can be difficult. For that reason, we often 'simplify' formulas by (a) removing some or all parentheses around negations, and (b) removing the outermost parentheses. Thus,

$$(\sim (\sim (\sim (\sim (\sim (\sim (\sim (\sim A))))))))$$

might be written

$$\sim\sim\sim\sim\sim\sim\sim\sim A$$

and

$$(((\sim A) \wedge (B \vee (\sim A))) \Rightarrow (\sim (\sim C)))$$

becomes

$$(\sim A \wedge (B \vee \sim A)) \Rightarrow \sim\sim C$$

This is *not* a WFF, just *shorthand* for one.

There is generally no ambiguity in interpretation, if you just imagine that the $\sim$ 'adheres' to the expression that follows immediately after it. You can always convert such an expression back to a WFF by finding expressions of the form $\sim P$ with no parentheses, add the parentheses back, then parenthesize the whole formula if necessary. For example,

$$\sim\sim A \Rightarrow \sim (B \wedge C)$$

becomes

$$\sim (\sim A) \Rightarrow \sim (B \wedge C)$$

then

$$(\sim (\sim A)) \Rightarrow \sim (B \wedge C)$$

then

$$(\sim (\sim A)) \Rightarrow (\sim (B \wedge C))$$

then

$$((\sim (\sim A)) \Rightarrow (\sim (B \wedge C)))$$

11.3 Semantics of Propositional logic:

Semantics (for any logic) refers to the assignation of meaning to the symbols.

In propositional logic, the 'meaning' of a formula is just a truth value.

Use **T** to represent 'True', **F** to represent 'False'

Consider a typical formula: $A \vee B$

We know that intuitively that the English sentence this represents, "Either A is true or B is true (or both)" is either true or false depending on the truth values of A and B. So, a formula won't be simply "true" or "false", but rather true or false relative to some fixed assignment of truth values to the most basic subformulas, the proposition letters.

Definition: A *model* for a propositional logic is an assignment of truth values to the proposition letters.

Remark: Synonyms for *model* are *interpretation* and *structure.*

Formally, if $\mathcal{B}$ is the set of proposition letters of our logic, then a model is a function $\mathcal{M} : \mathcal{B} \rightarrow \{\mathrm{T}, \mathrm{F}\}$.

Example: Suppose our logic has only the proposition letters A, B, and C. One model might be to make A true, and both B and C false. We might denote this by $\mathcal{M}(A) = T, \mathcal{M}(B) = F, \mathcal{M}(C) = F$. Or we could write: "$\mathcal{M}$ is the model $A = T, B = F, C = F$." Whatever is easier!

We already have rules telling us how to piece together the truth values of complex formulas from those for simpler ones. For example, with the model $\mathcal{M}$ we just defined, the formula

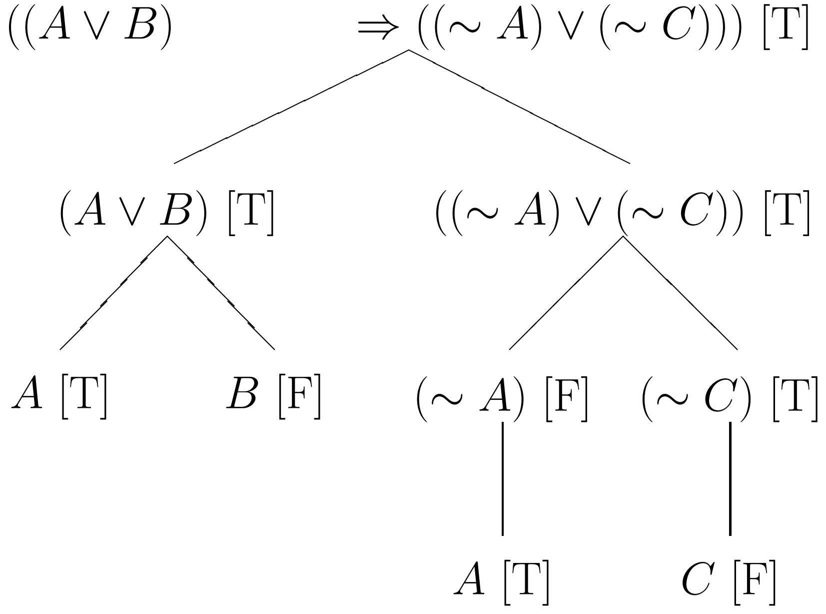
$$(A \vee (\sim (\sim B)))$$

would have the truth value T, since A is true and it is a disjunction of A with something.

For more complicated formulas, it is useful to use the same kind of syntactic breakdown we did before:

Example - using trees Recall again the formula: $((A \vee B) \Rightarrow ((\sim A) \vee (\sim C)))$

Here is the same tree as before, only we label the tree nodes (from the bottom-up) with truth values:



We can also do this using the other method of parsing formulas:

Recall: $\underbrace{(\underbrace{(A \vee B)} \Rightarrow \underbrace{(\underbrace{(\sim A)} \vee \underbrace{(\sim C)})})}$

Instead of underbracing, let's assign truth values, in the same order:

First: $((\underset{T}{A} \vee \underset{F}{B}) \Rightarrow ((\sim \underset{T}{A}) \vee (\sim \underset{F}{C})))$

Then: $((\underset{T}{A} \underset{T}{\vee} \underset{F}{B}) \Rightarrow ((\underset{F}{\sim} \underset{T}{A}) \vee (\underset{T}{\sim} \underset{F}{C})))$

Then: $((\underset{T}{A} \underset{T}{\vee} \underset{F}{B}) \Rightarrow ((\underset{F}{\sim} \underset{T}{A}) \underset{T}{\vee} (\underset{T}{\sim} \underset{F}{C})))$

Therefore: $((\underset{T}{A} \underset{T}{\vee} \underset{F}{B}) \underset{T}{\Rightarrow} ((\underset{F}{\sim} \underset{T}{A}) \underset{T}{\vee} (\underset{T}{\sim} \underset{F}{C})))$

Exercise: verify that in the model with A, B, and C all True, this formula is false.

Remark on *truth tables*

Based on this example, we can formally define what it means for a formula to be *true in a model*:

Definition of $\mathcal{M} \models \phi$ (ϕ is true in $\mathcal{M}$), where ϕ is a proposition and $\mathcal{M}$ is a model:

1. If ϕ is a proposition letter, then $\mathcal{M} \models \phi$ provided $\mathcal{M}(\phi) =$T.
2. If ϕ is $(\sim \psi)$ then $\mathcal{M} \models \phi$ provided it is *not* the case that $\mathcal{M} \models \psi$.
3. If ϕ is $(\psi \wedge \theta)$ then $\mathcal{M} \models \phi$ provided $\mathcal{M} \models \psi$ AND $\mathcal{M} \models \theta$.
4. If ϕ is $(\psi \vee \theta)$ then $\mathcal{M} \models \phi$ provided $\mathcal{M} \models \psi$ OR $\mathcal{M} \models \theta$.
5. If ϕ is $(\psi \Rightarrow \theta)$ then $\mathcal{M} \models \phi$ provided:
   IF $\mathcal{M} \models \psi$ THEN $\mathcal{M} \models \theta$
   (that is, *unless* $\mathcal{M} \models \psi$ but $\mathcal{M} \not\models \theta$).

Remark: Note the introduction of the notation $\models$ for "models"

Remark: We also used $\mathcal{M} \not\models \phi$ for "it is not the case that $\mathcal{M} \models \psi$"

A formula could be true in some models, not true in others.

In particular, *without specifying a model it makes no sense to say that a WFF is true or false!*

Some formulas are true in *every* model.

Example: $(A \vee (\sim A))$

Definition: A WFF ϕ is *valid* if it is true in every model, i.e., if $\mathcal{M} \models \phi$ for every assignment of truth values to proposition letters.