

11.3.2 Applications: Deduction Rules

Sections 3.2–3.4 and 3.6 of Gemignani introduce a number of *argument forms*.

These provide a template, where, from certain hypotheses (called here *premises*), a *conclusion* can be drawn.

Example: If I am a mathematician then I eat kittens. I am a mathematician. Therefore I eat kittens.

This is an example of *modus ponens*.

It can be represented in a general way as follows:

$$\begin{array}{l} A \Rightarrow B \\ A \\ \hline \therefore B \end{array}$$

Here $A \Rightarrow B$ and A are the premises, B is the conclusion.

A and B can just be proposition letters, or could stand for more complex well-formed-formulas.

When written in this “propositional logic” form, we call such an argument form a *deduction rule*.

You **will not** be required to know these deduction rules by name, or even at all (except maybe modus ponens, which every educated human should know by name).

You have already seen one other rule, *modus tollens*:

$$\begin{array}{l} A \Rightarrow B \\ \sim B \\ \hline \therefore \sim A \end{array}$$

You **will** be required to know how to verify that a deduction rule is *sound* (what the text calls a “valid argument”); I will illustrate this below, but also see text §3.2 et al.

Here are the steps, in general:

- Form a truth table containing all premises, the conclusion, and all subformulas of these formulas.
- Delete rows in which any premise fails, leaving all and only rows where every premise is true.
- If the conclusion is true in *all* of the remaining rows, then it is a sound deduction rule, otherwise it is not.

(Note the resemblance to the tennis example from last lecture!)

Example: Verify that modus ponens is sound:

A	B	$(A \Rightarrow B)$	
T	T	T	
T	F	F	← (delete this row)
F	T	T	← (delete this row)
F	F	T	← (delete this row)

Example: Verify that the following nameless deduction rule is sound:

$$\begin{array}{l}
 A \Rightarrow B \\
 \sim B \vee C \\
 \sim C \\
 \hline
 \therefore \sim A
 \end{array}$$

A	B	C	$\sim B$	$\sim B \vee C$	$\sim C$	$\sim A$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

Example: Is this deduction rule sound?

$$\begin{array}{l}
 A \Rightarrow B \\
 B \\
 \hline
 \therefore A
 \end{array}$$

A	B	$(A \Rightarrow B)$
T	T	T
T	F	F
F	T	T
F	F	T

A Lewis Carroll example

- a) All babies are illogical.
- b) Nobody is despised who can manage a crocodile.
- c) Illogical persons are despised.

Let proposition letters stand for propositions within the syllogism:

B= one is a baby

L= one is logical

M= one can manage a crocodile

D= one is despised

Carroll's words can now be translated into formulas:

a') $B \Rightarrow (\sim L)$

b') $M \Rightarrow (\sim D)$

c') $(\sim L) \Rightarrow D$

And (next page) a truth table:

B	L	M	D	$(\sim L)$	$(\sim D)$	$B \Rightarrow (\sim L)$	$M \Rightarrow (\sim D)$	$(\sim L) \Rightarrow D$
T	T	T	T					
T	T	T	F					
T	T	F	T					
T	T	F	F					
T	F	T	T					
T	F	T	F					
T	F	F	T					
T	F	F	F					
F	T	T	T					
F	T	T	F					
F	T	F	T					
F	T	F	F					
F	F	T	T					
F	F	T	F					
F	F	F	T					
F	F	F	F					

12 Propositional Logic – Syntactics again – Deductions

In this section I will show you a formal definition of *formal deduction* in propositional logic.

This is not in any of our texts.

You **will not** be required to do any formal deduction. Reason: truth tables are adequate for determining the validity of a formula for propositional logic.

Why am I showing this to you?

1. You can't use truth tables for other logics, such as first-order logic.
2. The definition of “deduction” is roughly the same for all usual logics.
3. This definition formalizes (and recapitulates) the notion of “direct proof” we discussed earlier in the course.
4. other reasons I can't think of now.

Constituents of a formal deduction:

Axioms: A collection \mathcal{A} of formulas, called the *axioms* of the logic. These should be tautologies; but - even more important - they should be formulas which one believes without reservation. They should also be reasonably easy to identify.

Deduction Rules: A set of one or more deduction rules; these should be sound (in the sense of the previous discussion).

Example: The following is a set of axioms I sometimes use when teaching more advanced courses in Logic:

a) $p \Rightarrow (q \Rightarrow p)$

b) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

c) $\sim\sim p \Rightarrow p$

(The actual axiom set would include any WFF that has one of these forms, for example, in the first one we could substitute $(A \vee (\sim B))$ for p and $(C \Rightarrow A)$ for q and get the axiom

$$(A \vee (\sim B)) \Rightarrow ((C \Rightarrow A) \Rightarrow (A \vee (\sim B)))$$

For deduction rules I allow just one, Modus Ponens.

Definition of formal deduction: Let S be a set of formulas (called *premises*) and P is a proposition (called the *conclusion*). A *deduction* of P from S is a sequence of propositions such that each one is either an axiom, a premise, or follows from earlier propositions in the sequence using the rule(s) of inference.

More formally, a deduction of P from S is a sequence P_0, P_1, \dots, P_n of propositions such that:

- i) $P_n = P$
- ii) For each P_i , either
 - (a) $P_i \in \mathcal{A}$, or
 - (b) $P_i \in S$, or
 - (c) P_i follows from earlier propositions in the sequence using the rule(s) of inference.

Remarks:

This should remind you of our earlier notion of “direct proof,” on which it is based.

It is purely syntactic - we do no interpreting of the symbols in the definition (though of course interpretation might direct our choice of axioms and deduction rules).

The more axioms and deduction rules we have the ‘easier’ it is to be a deduction, but we usually keep the axiom set as small as possible, similar for the deduction rules; this is useful in practice.

If every axiom is a tautology, and the deduction rules are all sound, then we might conjecture that every proposition P for which there is a formal deduction follows from the premises in the semantic sense we discussed earlier. This is true (and can be proved). In particular, if our set S of premises is empty, then P is valid. This is sometimes called the *Soundness Theorem* of propositional logic.

Conversely, it is useful to have enough axioms and deduction rules so that every tautology can be proved deductively. (This *is* the case for the axioms above, with Modus Ponens.) This is called having a *complete* logic.