13 Completeness Theorem for Propositional Logic

(Assume we are working in a system with the axioms and deduction rule I gave above.)

Theorem 13.1. *Let S be a set of WFFs, and let P be a WFF; then the following are equivalent:*

1. *Every model of all the WFFs in S is a model of P* $\longleftarrow$ *semantic*
2. *There is a deduction of P from S* $\longleftarrow$ *syntactic*

The implication $1 \Rightarrow 2$ says that the logic is adequate to prove everything you would want to prove, including tautologies; that is why it is called a *completeness* theorem

The implication $2 \Rightarrow 1$ is the *Soundness Theorem* mentioned earlier, and says that you can't give a formal deduction of something false.

14 Sets

- A *set* is a collection of objects.
- This seems to be a vague and circular definition: what is an object? what is a collection?
- It *is* too vague; see Russell's paradox below.
- We'll worry about this later (though not much), not now.

How do we indicate a set?

By explicitly listing all elements:

$\{1, 2, 3\}$;
$\{a, b, Harvey\}$
$\{\}$

By implicitly listing all elements:

$\{1, 2, 3, \ldots, 1000\}$;
$\{a, b, \ldots, z\}$;
{Oahu, Maui, Kauai,…}

By name: $\mathbb{R}$; $\mathbb{N}$; $\mathbb{Q}$; $\emptyset$

By description, using set-builder notation:

$\{n \in \mathbb{N} \mid n \text{ prime}\}$;
$\{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N}\}$;
$\{n \in \mathbb{N} \mid n \neq n\}$

Note the use of the symbol ϵ, meaning "in" or "element of."

This is the only 'primitive' set operation; everything else will be defined in terms of ϵ.

We will say synonymously: $x \in A$; "x is an element of A"; "A contains x"

Russell's Paradox:

Consider the following definition of a set:

$$W = \{x \mid x \notin x\}$$

Suppose $W \notin W$.

Then W satisfies the definition of W, so $W \in W$.

So W must be in W, $W \in W$.

But to be in W an object must not contain itself, so $W \notin W$.

Uh oh.

The real conclusion to the above *paradox* is that the formula we gave above does not properly define a set; there is no "set of all sets."

So, how do we know whether the things we write down are really sets?

The Modern Approach: wrote down formal properties for sets corresponding to our intuition, try not to include enough properties to construct something paradoxical.

I will show you these axioms later, for right now we will focus on basic set operations and try not to worry too much whether the the things we have written down are actually sets. However, the defining properties should always be clear enough that *if* the thing we define is a set, *then* there is no doubt what the elements of that set would be. (cf Gem. §4.1)

Notation and operations:

Extensionality: A set is completely determined by its contents, not how that set is presented. So, the following sets are all equal:

1. $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
2. $\{5, 4, 3, 2, 1, 6, 7, 8, 9\}$ (order doesn't matter)
3. $\{1, 1, 1, 1, 1, 2, 3, 4, 5, 4, 3, 2, 1, 9, 6, 7, 8, 9\}$ (duplication doesn't matter)
4. $\{1, 2, 3, \ldots, 9\}$ (as long as the ellipsis is unambiguous)
5. $\{x \in \mathbb{N} \;:\; 1 \leq x \leq 9\}$ (but note how important specifiying the $\mathbb{N}$ is!)

When we write $A = B$ (where A and B are sets) we mean the following synonymous things:

1. Every element of A is an element of B and vice versa.
2. $\forall x\ (((x \in A) \Rightarrow (x \in B)) \wedge ((x \in B) \Rightarrow (x \in A)))$