14.4 Venn Diagrams, Subsets, and Predicates

Suppose we have a univers of discourse (or *universal set*) X.

A subset of X can be specified by collecting together all elements satisfying some property.

Example: X = all students at UHM, $A = \{x \in X \mid x \text{ is an English major } \}$

We could write $E(x)$ as shorthand for "x is an English major"

In this notation, $A = \{x \in X \mid E(x)\}$

Written this way, such a property is sometimes called a *predicate*.

Set-builder notation reflects the following axiom, sometimes called the Axiom of Collection (or the Axiom of Subset Selection):

> If $P(x)$ is a predicate, then there is a set consisting exactly of those elements x of X for which $P(x)$ holds.
>
> In other words, $\{x \in X \mid P(x)\}$ exists.

It is common (if confusing!) to use $A(x)$ for the predicate used to select A.

Note that for any given x, $P(x)$ has a truth value; we can therefore translate statements about relations and operations on sets into logical statements involving predicates.

Examples:

1. "All men are mortal."
   $M(x) = x$ is a man
   $D(x) = x$ is mortal
   Then the assertion becomes:

$$\forall x(M(x) \Rightarrow D(x))$$

   If $M = \{x \in X \mid M(x)\}$, $D = \{x \in X \mid D(x)\}$; then the above becomes

$$(M \subset D)$$

   (What is a good candidate for X?)

2. More generally, if

   $A = \{x \in X \mid A(x)\}$ and $B = \{x \in X \mid B(x)\}$ then

$$(A \subset B) \text{ is } \forall x(A(x) \Rightarrow B(x))$$

3. Another Lewis Carroll example:

   (a) None of the unnoticed things, met with at sea, are mermaids.
   (b) Things entered in the log, as met with at sea, are sure to be worth remembering.
   (c) I have never met with anything worth remembering, when on a voyage.
   (d) Things met with at sea, that are noticed, are sure to be recorded in the log.

   X =things you might meet at sea
   $N(x) = x$ is noticed
   $M(x) = x$ is a mermaid
   $L(x) = x$ is entered in the log
   $R(x) = x$ is worth remembering
   $I(x) =$ I have met with x at sea
   So:

   (a) $\forall x((\sim N(x)) \Rightarrow (\sim M(x)))$

(b) $\forall x(L(x) \Rightarrow R(x))$

(c) $\forall x(I(x) \Rightarrow \sim R(x))$

(d) $\forall x(N(x) \Rightarrow L(x))$