11 Propositional (or *sentential*) Logic

11.1 Elements of the formal language; Syntax

Symbols:

- Proposition Letters: $A\ B\ C\ \cdots$ (or sometimes $A_0\ A_1\ \cdots$)
- Logical Connectives: $\wedge\ \vee\ \implies\ \sim$
- Other ("non-logical") Symbols: ( )

Formal Definition of a *proposition* (or *sentence*, or *well-formed formula* [abbreviated WFF]):

1. If P is a proposition letter then P is a proposition.
2. If P is a proposition then $(\sim P)$ is a proposition.
3. If P and Q are propositions then $(P \wedge Q)$, $(P \vee Q)$, and $(P \Rightarrow Q)$ are propositions.
4. Nothing is a proposition unless defined via rules (1)-(3).

The importance of this formal definition is (a) it gives a mechanical way to 'recognize' a proper formula, and (b) that such a formal definition can be done at all!

Remark: The definition of WFF is an example of a *recursive definition*; it explains how to generate complex propositions from simpler ones:

Simplest: $A, B, \ldots$

Next: $(\sim\ A), (\sim\ B), (A \wedge A), (A \wedge B), (B \wedge B), (A \vee A), (A \vee B), (B \vee B), (A \Rightarrow A), (A \Rightarrow B), (B \Rightarrow B), \ldots$

Next: $(A \wedge (\sim B)), ((A \wedge A) \Rightarrow (A \wedge B)), \ldots$

etc.

11.2 How to recognize a Well-Formed Formula

The definition suggests several ways to identify if a formula is well-formed. One is a *bottom-up* approach, another is *top-down*

Bottom-up Identify constituent formulas (called *subformulas*) in *increasing* order of complexity.

Example: $((A \vee B) \Rightarrow ((\sim A) \vee (\sim C)))$

The proposition letters (in this case A, B, and C) are the simplest subformulas.

Find innermost paired parentheses, they delineate the next simplest subformulas:

$$(\underbrace{(A \vee B)} \Rightarrow (\underbrace{(\sim A)} \vee \underbrace{(\sim C)}))$$

Treat identified subformulas as single units, and look again for paired parentheses:

$$(\underbrace{(A \vee B)} \Rightarrow \underbrace{(\underbrace{(\sim A)} \vee \underbrace{(\sim C)})})$$

And again:

$$\underbrace{(\underbrace{(A \vee B)} \Rightarrow \underbrace{(\underbrace{(\sim A)} \vee \underbrace{(\sim C)})})}$$

Example: The following is *not* a WFF:

$((A \vee B) \Rightarrow ((\sim A)(\sim C)))$

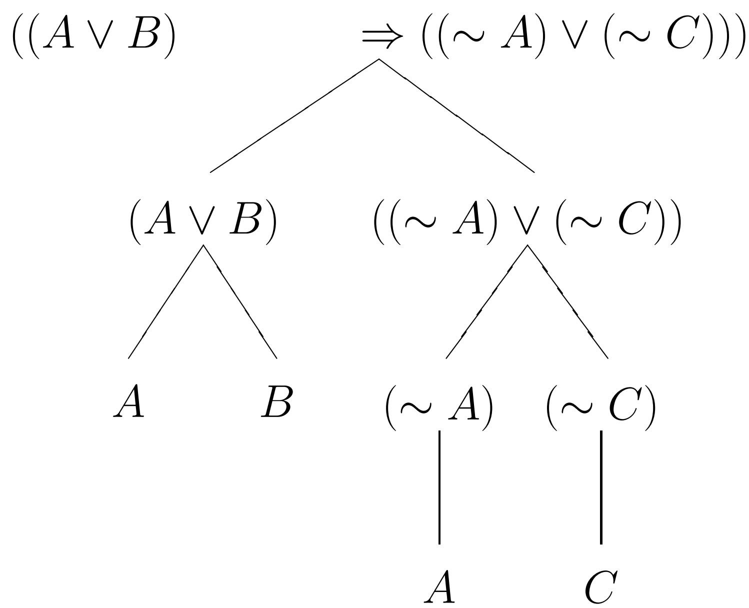
The process above makes it this far:

$$(\underbrace{(A \vee B)} \Rightarrow (\underbrace{(\sim A)}\underbrace{(\sim C)}))$$

But now we're stuck, since we see two adjacent subformulas that are not connected with a logical connective.

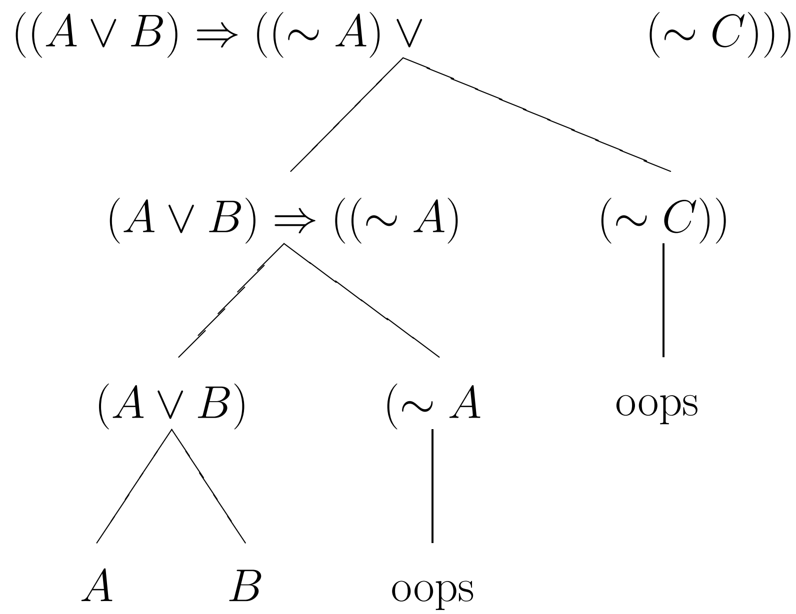
Top-down: Identify constituent formulas (called *subformulas*) in *decreasing* order of complexity.

Example: $((A \vee B) \Rightarrow ((\sim A) \vee (\sim C)))$



Warning: You *must* correctly parse at each level!

Example: (again) $((A \vee B) \Rightarrow ((\sim A) \vee (\sim C)))$



How do you avoid getting lost in the parentheses? Count!

Add one for each left parenthesis, subtract one for each right parenthesis:

$$\underset{12}{((}A \vee B\underset{1}{)} \Rightarrow \underset{23}{((} \sim A\underset{2}{)} \vee \underset{3}{(} \sim C\underset{210}{)))}$$

The first main subformula will stretch from the first 2 to the first 1 after that 2.

More examples (class).