9.2 Negation

Suppose A is a propositions. The following are all different ways of saying the same thing:

$$A \text{ is false.}$$
$$A \text{ is not true.}$$
$$\text{Not } A.$$
$$\sim A$$

Intuitively, $(\sim A)$ has the opposite truth value of A, and is called a *negation* of A. Given a statement, you can always find a negation by putting "not" in front of it or "is false" after it. However, recognizing that one statement is a negation of another is not always easy - in math, or in life.

Examples

1. The moon is made of green cheese.

   Some Negations:

   (a) The moon is not made of green cheese.

   (b) It is false that the moon is made of green cheese.

   (c) The moon fails to be made of green cheese.

   Some Invalid Negations

   (a) The moon is made of poi.

   (b) Mars is made of green cheese.

One way to recognize that B is a negation of A is:

If A is true then B must be false.

If A is false then B must be true.

If one or both of the "must be" statements is not a real "must" then B is not a negation of A.

2. Which are negations of $1 + 1 = 2$?

(a) $1 + 1 \neq 2$

(b) $1 + 1 = 0$

The question of whether $1 + 1 = 0$ is a negation of $1 + 1 = 2$ is an extremely subtle one. They certainly have opposite truth values, but one can imagine a universe in which they are both true. (Can't you? How about $6 + 6 = 0$? Hint: clock arithmetic.)

We will generally only use the term *negation* when referring to "syntactic" opposites.

3. Which are negations of "All mathematics professors eat kittens"?
   (a) No mathematics professors eat kittens.
   (b) Some mathematics professors don't eat kittens.
   (c) All mathematics profeesors eat only dogs.
   (d) David Ross doesn't eat kittens.

Negating sentences beginning with "All" or "Every" or $\forall x$:

A negation of "$\forall x\ \Phi(x)$" is "$\exists x(\sim \Phi(x))$"

(Recall that $\exists x \cdots$ means "there exists an x such that $\cdots$" or "Some x are $\cdots$" or "For at least one $x, \cdots$", etc.)

4. Find negations:
   (a) Every good boy does fine.
   (b) All politicians are liars.
   (c) $\forall N\ 5|N$.
   (d) No mathematics professors eat kittens.

So - to prove that a 'for all x" statement is *false*, you only need to find *one* x that makes it false. This is sometimes called *Disproof by Counterexample.*

5. What is a negation of "If $3|N$ then N is composite"?

   (**Answer:** "$3|N$ and N is prime.")

Note that the only way $A \implies B$ can be *false* is when A is true but B is false. To negate $A \implies B$ we want something which is only *true* in this situation, and the obvious statement is then "$A \wedge \sim B$".

Two more remarks about negation:

1. $\sim\sim A$ is equivalent to A
2. Recall that
$$A \implies B$$
is equivalent to
$$(\sim B) \implies (\sim A)$$
(contrapositive)