

16.2 Complex Numbers

Recall: The *complex numbers* \mathbb{C} are obtained by adding the ‘ideal’ element $i = \sqrt{-1}$ to the real numbers.

That is, $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Addition, multiplication et al are done formally, as if the i was a variable like x , but using the simplifying rule $i^2 = -1$

- Examples:**
1. $(3 + 2i) + (1 + i) = ?$
 2. $(3 + 2i) \times (1 + i) = ?$
 3. $\frac{(3+2i)}{(1+i)} = ?$ (Harder!)

Note that in each case, the result could be put in the form $a + bi$ again. More generally,

\mathbb{C} is closed under addition, subtraction, multiplication, and division.

A number of the form ai (i.e., $0 + ai$) is called *imaginary*

Since imaginary numbers are “made up”, it is reasonable to ask whether operations involving them are sensible.

One way to give them more meaning is to find a concrete model for them.

Note that $(a, b) \leftrightarrow a + bi$ establishes a “one-to-one correspondence” between complex numbers and ordered pairs of real numbers (=points in the plane).

We could thus *define* a complex number to be a point in the plane, and then define operations accordingly; see the handwritten notes, Stein, and Stewart.

The following is very important, and very hard!

Fundamental Theorem of Algebra:

If $p(x) = ax^n + \dots$ is an n th degree polynomial, then p factors completely as

$$p(x) = a(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n)$$

where r_1, \dots, r_n are roots of the equation

$$p(x) = 0.$$

This result holds even if the coefficients of $p(x)$ are allowed to be complex numbers, not just integers or even reals.

In other words, we never have to leave \mathbb{C} to do any of the ordinary operations of algebra, including solving polynomial equations.

16.3 Axioms

Recall the Peano Axioms for \mathbb{N} :

1. $\forall x (0 \neq s(x))$
2. $\forall x \forall y ((x \neq y) \Rightarrow (s(x) \neq s(y)))$
3. $\forall x ((x \neq 0) \Rightarrow \exists y (x = s(y)))$
4. For *any* predicate $P(x)$, we have the axiom:

$$((P(0) \wedge \forall x (P(x) \Rightarrow P(s(x)))) \Rightarrow \forall x P(x))$$

What are the elements of the language (predicate logic) used here?

Variables: $x, y, z, \dots; x_0, x_1, x_2, \dots$

Equality symbol: $=$

Logical connectives: $\wedge, \vee, \sim, \Rightarrow$

Quantifier symbols: \forall, \exists

Punctuation: $), ($

All the above are the same for *any* predicate logic

Especially for discussing \mathbb{N} we introduced:

A **constant symbol** 0 ;

A **function symbol** s ;

Sometimes we call $\{0, s\}$ the *language of* \mathbb{N} .

To specify a model for “Peano Arithmetic” (= the above axioms in this language), we need to specify a universe of discourse and interpretations of the symbols 0 and s .

For example, the *natural model* \mathcal{N} for Peano Arithmetic has:

Domain of discourse is \mathbb{N}

0 is interpreted by the natural number 0

s interpreted by the function (defined on \mathbb{N})

$$s(x) = x + 1$$

A formal definition of “ P is true in \mathcal{N} ” (or even of “ P is a well-formed formula”) is too hard for this class, we will just work with the intuitive meaning as we have been doing all semester.

We can now write axioms for other systems.

Integers:

Constant symbols $0, 1$; function symbols $+, \times$

Axioms for \mathbb{Z} (corrected 3/28):

1. $\forall x \forall y \forall z ((x + y) + z = x + (y + z))$
2. $\forall x ((x + 0 = x) \wedge (0 + x = x))$
3. $\forall x \exists y ((x + y = 0) \wedge (y + x = 0))$
4. $\forall x \forall y (x + y = y + x)$
5. $\forall x \forall y \forall z ((x \times y) \times z = x \times (y \times z))$
6. $\forall x ((x \times 1 = x) \wedge (1 \times x = x))$
7. $\forall x \forall y \forall z ((x + y) \times z = (x \times z) + (y \times z))$
 $\forall x \forall y \forall z ((z \times (x + y) = (z \times x) + (z \times y))$
8. $\forall x \forall y (x \times y = y \times x)$

Remark: Definition of *group*, *abelian*, *ring*.