We can now write axioms for other systems.

Integers:

Constant symbols $0, 1$; function symbols $+, \times$

Axioms for $\mathbb{Z}$ (corrected 3/28):

1. $\forall x \forall y \forall z\ ((x + y) + z = x + (y + z))$
2. $\forall x\ ((x + 0 = x) \wedge (0 + x = x))$
3. $\forall x \exists y\ ((x + y = 0) \wedge (y + x = 0))$
4. $\forall x \forall y\ (x + y = y + x)$
5. $\forall x \forall y \forall z\ ((x \times y) \times z = x \times (y \times z))$
6. $\forall x\ ((x \times 1 = x) \wedge (1 \times x = x))$
7. $\forall x \forall y \forall z\ ((x + y) \times z = (x \times z) + (y \times z))$
   $\forall x \forall y \forall z\ ((z \times (x + y) = (z \times x) + (z \times y))$
8. $\forall x \forall y\ (x \times y = y \times x)$

Remark: Definition of *group*, *abelian*, *ring*.

Rationals:

Constant symbols $0, 1$; function symbols $+, \times$

Axioms for $\mathbb{Q}$:

1. $\forall x \forall y \forall z \; ((x + y) + z = x + (y + z))$
2. $\forall x \; ((x + 0 = x) \wedge (0 + x = x))$
3. $\forall x \exists y \; ((x + y = 0) \wedge (y + x = 0))$
4. $\forall x \forall y \; (x + y = y + x)$
5. $\forall x \forall y \forall z \; ((x \times y) \times z = x \times (y \times z))$
6. $\forall x \; ((x \times 1 = x) \wedge (1 \times x = x))$
7. $\forall x \; (x \neq 0 \Rightarrow (\exists y ((x \times y = 1) \wedge (y \times x = 1))))$
8. $\forall x \forall y \forall z \; ((x + y) \times z = (x \times z) + (y \times z))$
   $\forall x \forall y \forall z \; ((z \times (x + y) = (z \times x) + (z \times y))$
9. $\forall x \forall y \; (x \times y = y \times x)$

Reals:

Constant symbols $0, 1$; function symbols $+, \times$

Axioms for $\mathbb{R}$:

1. $\forall x \forall y \forall z \ ((x + y) + z = x + (y + z))$
2. $\forall x \ ((x + 0 = x) \wedge (0 + x = x))$
3. $\forall x \exists y \ ((x + y = 0) \wedge (y + x = 0))$
4. $\forall x \forall y \ (x + y = y + x)$
5. $\forall x \forall y \forall z \ ((x \times y) \times z = x \times (y \times z))$
6. $\forall x \ ((x \times 1 = x) \wedge (1 \times x = x))$
7. $\forall x \ (x \neq 0 \Rightarrow (\exists y((x \times y = 1) \wedge (y \times x = 1))))$
8. $\forall x \forall y \forall z \ ((x + y) \times z = (x \times z) + (y \times z))$
   $\forall x \forall y \forall z \ ((z \times (x + y) = (z \times x) + (z \times y))$
9. $\forall x \forall y \ (x \times y = y \times x)$
10. LUB axiom(s)

Note: to formulate the LUB axioms, we really need to discuss *order*. The usual ordering on the reals is definable in the language:

$$x < y \leftrightarrow \exists z \ (x + z^2 = y)$$

Thus, we might as well assume that the language has the order symbol as part of the syntax.

In English, the LUB property was: Every set of real numbers which is bounded above has a least upper bound

In predicate logic: for every predicate $P(x)$, add the ugly-looking axiom

$$(\exists y \forall x (P(x) \Rightarrow (x < y))) \ \Rightarrow$$
$$(\exists y (\forall x (P(x) \Rightarrow (x < y))) \wedge (\forall z ((z < y) \Rightarrow$$
$$(\exists w (z < w) \wedge P(w)))))$$

Unfortunately, this only gives the LUB property for sets that correspond to a predicate $P(x)$.

This turns out not be enough; in fact, there is a set E with the property:

1. $\mathbb{Q} \subsetneq E$
2. $E \subsetneq R$
3. E is closed under $+, \times$
4. E satisfies all the axioms of $\mathbb{Q}$ (the *field axioms*)
5. E satisfies all the LUB axioms corresponding to predicates.

We can even have a nested sequence of such sets:

$$\mathbb{Q} \subsetneq E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq \mathbb{R}$$

This inability to pin down the reals with first-order logic was just discovered in the 20th century.

On the other hand:

$\mathbb{R}$ is the UNIQUE field extension of $\mathbb{Q}$

satisfying the LUB property for all subsets.

"Unique" here means "unique up to *isomorphism*" – whatever that means!

(Uhh...what's the difference again?)

Complex:

$\mathbb{C}$ is the smallest field extending $\mathbb{R}$ such that the equation $x^2 = -1$ has a solution.

Note that it is hard to formally retain the LUB axioms, since the $<$ is no longer easily definable in $\mathbb{C}$.

We won't even try to find axioms that completely charactize $\mathbb{C}$.

For working axioms, just take the axioms for $\mathbb{Q}$ together with the axiom $\exists x(x^2 + 1 = 0)$

Other: There are many other ‘reasonable’ extensions of $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$

Examples: 1. Quaternions (for mechanics)

2. Computable real or compex numbers

3. Infinitesimals (Calculus)

Infinitesimals:

Idea: a *positive infinitesimal* is a ‘number’ $\epsilon > 0$ such that $0 < \epsilon$ but $\epsilon < 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$

Archimedes derivation of area of circle.