Rationals and Repeating Decimals

Sometimes the digits in the decimal expansion of a number start repeating after a while.

Examples: 1. $0.3333333\ldots$

2. $1.17171717171717\ldots$

3. $47.1973402340234023\ldots$

4. -12.45

Note: all the above are rational numbers

Example: $0.99999\cdots = 1$

Theorem $\forall x$ (x is a repeating decimal $\Rightarrow x \in \mathbb{Q}$)

Equivalently: $\forall x$ (x irrational $\Rightarrow x$ not a repeating decimal)

Proof: Clearly we can do to *any* repeating decimal what we did in the examples above.

Question: Is the converse true?

Is every number whose decimal representation does *not* repeat an irrational number?

Equivalently, does every rational number have a repeating decimal representation?

Theorem Yes.

(See Stein for the proof.)

Examples: 1. 47/2

2. 47/5
3. 47/25
4. 47/15
5. 12/7

Some Famous Irrational Numbers

$\sqrt{2}$

$\sqrt{p}$ (for any prime p)

π $(= 3.14159265358979323846\ldots)$

e $(= 2.71828182845904523536028747135266249775 72\ldots)$

ϕ (Phi, the *Golden ratio*) The positive solution to the following remarkable quadratic equation:

$$\phi^2 - \phi - 1 = 0$$

$(= 1.6180339887498948482045868343656381177203 09\ldots)$ (but what is it *exactly*?)

Any number of the form $p+qz$ where z irrational, $p, q \in \mathbb{Q}$ and $q \neq 0$

$0.1010010001000010000010000001000000001\ldots$

$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots$ ("Zeta of 3")

$(= 1.2020569031595942853997381615114499076\ldots)$

$\zeta(n) = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \cdots$ is *Riemann's Zeta Function*, and is connected to one of the most important open problems in mathematics today (the *Riemann Hypothesis*).

That $\zeta(3)$ is irrational has only been known since 1979 (story)

We still don't know many things about $\zeta(3)$, including whether it is *algebraic* (next lecture)

16.1 Algebraic numbers

Two of our main examples were irrational because they solved simple equations:

$$\sqrt{2} \text{ satisfies } x^2 = 2$$

$$\phi \text{ satisfies } \phi^2 - \phi - 1 = 0$$

Question: Is *this* why we need the reals? To be able to solve equations?

Recall: A *polynomial* is a function that looks like:

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

$a_0, a_1, \ldots, a_n$ are the *coefficients* of the polynomial. Usually assume the *leading coefficient* $a_n \neq 0$

Question: If $p(x)$ is a polynomial, when does the equation $p(x) = 0$ have a solution?

$p(x) = ax + b$ (*linear* polynomial), then always.

$p(x) = ax^2 + bx + c$ (*quadratic* polynomial), then

$$p(x) = 0$$

is 'solved' by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What about cubic? quartic? quintic?