Definition: A number is *algebraic* if it is a solution to an equation of the form $p(x) = 0$, where $p(x)$ is a polynomial with integer coefficients.

A real number is *transcendental* if it is not algebraic.

Examples: 1. $-13/17$

2. $\sqrt{2}$; ϕ; $2^{\frac{1}{4}}$

3. $\sqrt{3-\sqrt{2}}$

4. $\sqrt{5}+\sqrt{7}$

So....why $\mathbb{R}$?

Start with a 'typical' real number, say $\pi = 3.1415926\ldots$

Approximate: $3, 3.1, 3.14, 3.141, 3.1415, \cdots$

Note that these rational approximations are *increasing*, and *bounded above* (by 4, for example).

Another example (*continued fraction*):

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}$$

Approximate: $1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{1+1}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}, \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}, \ldots$

These rational approximations are *increasing*, and *bounded above* (by 1, for example).

This motivates the following operational definition:

The set $\mathbb{R}$ of *real numbers* is the smallest set which contains $\mathbb{Q}$ and satisfies the following property, the *least upper bound* (LUB) property:

If $\emptyset \not\subset A \subset \mathbb{R}$ and A is bounded above then there is a number a such that

(i) $\forall x(x \in A \Rightarrow x \leq a)$

(ii) $\forall y((\forall x(x \in A \Rightarrow x \leq y)) \Rightarrow a \leq y)$

Example $\frac{1}{1+\frac{1}{1+\frac{1}{1+\dots}}}$ must be a real number. What number is it?

Problem: Even in $\mathbb{R}$ we can't solve some simple equations, such as:

$$x^2 + 1 = 0$$

Solution: Extend the system yet again, to the *Complex Numbers* $\mathbb{C}$

$\mathbb{C}$ will be is the smallest extension of $\mathbb{R}$ that satisfies the usual algebraic properties of $\mathbb{R}$ and also contains an 'imaginary' element representing $\sqrt{-1}$.

Remarkably, just throwing $\sqrt{-1}$ into the system makes it possible to solve *all* polynomial equations; this is the *Fundamental Theorem of Algebra.*