4. “All men are mortal; Socrates is a man; therefore Socrates is mortal”
   $M(x) = x$ is a man
   $D(x) = x$ is mortal
   $s =$Socrates

$$\begin{array}{l} \forall x\ (M(x) \Rightarrow D(x)) \\ \underline{M(s) \qquad\qquad} \\ \therefore D(s) \end{array}$$

5. "All cats die. Socrates is dead. Therefore Socrates is a cat."
$C(x) = x$ is a cat
$D(x) = x$ dies
$s =$Socrates
$\forall x\ (C(x) \Rightarrow D(x))$

$$\frac{D(s)}{\therefore C(s)}$$

Existential statements

6. "Some birds are red."
   $B(x) = x$ is a bird
   $R(x) = x$ is red
   $\exists x \; (B(x) \wedge R(x))$

   Equivalent:

   "Some red things are birds."
   "there exists at least one red bird."

15 Predicate Logic

The formulas from the last couple of sections - formulas like those we have been using all semester - are examples of Well-Formed Formulas in *Predicate Logic* (sometimes called *First-order Logic*).

Reason for the name: this logic is expressive enough to let us discuss set membership, subset, and operations like intersection and union.

It is also expressive enough to formalize most of mathematics.

For example, here is a formal definition of the Natural Numbers; it uses two special symbols, a symbol for 0 and a symbol s where $s(x)$ is meant to be the 'successor' of x (i.e., $x+1$):

Peano Postulates

1. $\forall x\ (0 \neq s(x))$
2. $\forall x \forall y\ ((x \neq y) \Rightarrow (s(x) \neq s(y)))$
3. $\forall x\ ((x \neq 0) \Rightarrow \exists y\ (x = s(y)))$
4. For *any* predicate $P(x)$, we have the axiom:

$$((P(0) \wedge \forall x(P(x) \Rightarrow P(s(x))) \Rightarrow \forall x P(x))$$

Remarkably, this is enough to do anything we might want to do in $\mathbb{N}$!

Examples:

Any natural number can be represented, for example 17 is represented by

$$s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(0)))))))))))))))))$$

Addition can be defined in terms of successor:

$x + 0 =_{def} x$

$x + s(y) =_{def} s(x + y)$

For example, $1 + 1 = 2$:

$2 + 2 = 4$: