The John von Neumann Natural Numbers

To use the Peano Postulates we really need to know that these formulas are consistent. The easiest way to show that they *are* is to build a 'model' with familiar objects that satisfy these postulates.

J. von Neumann (1903-1957) proposed the following model:

All the elements of this model are sets.

For 0 we take the emptyset, $0 =_{def} \emptyset$

For s we take the function $s(x) =_{def} x \cup \{x\}$

So:

$0 = \emptyset = \{\}$

$1 = s(0) = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}$

$2 = s(1) = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\}$

(or - if you prefer - $\{\emptyset, \{\emptyset\}\}$)

$3 = s(2) = 2 \cup \{2\} = \{0, 1\} \cup \{2\} = \{0, 1, 2\}$

(or - if you prefer - $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$)

More Generally:

$n+1 = n \cup \{n\} = \{0, 1, 2, \ldots, n-1\} \cup \{n\} = \{0, 1, 2, \ldots, n\}$

Every "von Neumann" natural number is the set of its predecessors.

Note that the for any m and n in this model,

$$m < n \text{ if and only if } m \in n$$

(convince yourself that this is true!), moreover in this case

$$m \subset n.$$

16 The hierarchy of number systems

God made the integers, all else is the work of man.

– Leopold Kronecker (1823-1891)

- Start with 0
- As soon as we can add 1 (the successor function) we get all of $\mathbb{N}$
- In $\mathbb{N}$ we can define addition and multiplication, but we can't subtract (without sometimes leaving $\mathbb{N}$)
- So: to get subtraction we move to $\mathbb{Z}$, the integers. (Axioms and model later.)
- In $\mathbb{Z}$ we can add, subtract, and multiply.
- $\mathbb{Z}$ is the smallest set of numbers extending $\mathbb{N}$ in which we can do this.
- However in $\mathbb{Z}$ we can't divide. (For example $5 \div 3$ is not an integer.)

- So: to get division we move to $\mathbb{Q}$, the rational numbers. (Axioms and model later.)
- In $\mathbb{Q}$ we can add, subtract, multiply, and divide (except by 0).
- $\mathbb{Q}$ is the smallest set of numbers extending $\mathbb{Z}$ in which we can do this.

Question: What can't we do in $\mathbb{Q}$?

Answer (if you're an electronic calculator): Nothing.

(examples in class)

Answer (if you're not a calculator): Plenty

Example: (earlier in semester) $\sqrt{2}$ is irrational (="not rational").

(Equivalently: there is no rational number whose square is 2.)

(Pythagoras: "The side of a square and its diagonal are not commensurate.")

Stein, Chapter 4: Other examples of irrational numbers obtained the same way.

Ubiquity of rationals: Between any two distinct numbers (rational or irrational) there is another rational number. (That is, $\mathbb{Q}$ is *dense*.)

Ubiquity of irrationals: Between any two distinct numbers there is an irrational number. (That is, $\mathbb{R} - \mathbb{Q}$ is dense.)