Complex Numbers

Def A <u>complex number</u> is an ordered pair (a,b) of real numbers.

Notation $\mathbb{C}$ = set of complex numbers

Write $a+bi$ for (a,b)

a = "real part" of $a+bi$

b = "imaginary part"

Identify $\mathbb{R}$ with $\{(a,0) \mid a \in \mathbb{R}\}$
$= \{a+0i \mid a \in \mathbb{R}\}$

EG: $3+2i$ for $(3,2)$
$3 = 3+0i$ for $(3,0)$
$0 = 0+0i$ for $(0,0)$
$i = 0+1i$ for $(0,1)$
etc

Properties

(1) $a+bi = c+di$ if & only if $a=c$ and $b=d$

(2) $i^2 = -1$

(3) 0 is an additive identity on $\mathbb{C}$; 1 is a multiplicative identity on $\mathbb{C}$

(4) Associative, Distributive, Commutative laws hold for $+, \times$ on $\mathbb{C}$

(5) $\mathbb{C}$ is closed under $+, \times, \div, \sqrt{}$

<u>Define</u> operations $+, \times$ on $\mathbb{C}$ by:

$$(a,b)+(c,d) = (a+c,\, b+d) \qquad (a+bi)+(c+di) = (a+c)+(b+d)i$$

$$(a,b)\times(c,d) = (ac-bd,\, ad+bc) \qquad (a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

<u>Note</u> multiplication is done by treating i like 'normal' number whose square is -1, and simplifying:

$$(a+bi)(c+di) = ac + adi + bci + bd\underbrace{i^2}_{=-1} = (ac-bd) + (ad+bc)i$$

<u>EG</u> $(3+2i)+(1+i) = 4+3i$

$(3+2i)(1+i) = 3+3i+2i-2 = 1+5i$

$\frac{3+2i}{1+i}$ = that $a+bi$ such that $(3+2i) = (1+i)(a+bi)$

$\therefore\ 3+2i = (a-b)+(a+b)i$

$\left.\begin{array}{l} 3 = a-b \\ 2 = a+b \end{array}\right\} \Rightarrow 2a = 5 \ \therefore a = 5/2,\ b = \frac{5}{2}-3 = -\frac{1}{2}$

$\therefore$ quotient $= \frac{5}{2} - \frac{1}{2}i$

$\sqrt{1+i} = ?$

$(a+bi)^2 = 1+i$

$(a^2-b^2) + 2abi = 1+i$

$\left.\begin{array}{l} a^2-b^2 = 1 \\ 2ab = 1 \end{array}\right\} \Rightarrow a = \pm\left(\frac{\sqrt{2}+1}{2}\right)^{\frac{1}{2}},\ b = \left(\frac{\sqrt{2}-1}{2}\right)^{\frac{1}{2}}$ (Show!)

$\therefore \sqrt{1+i} = \pm\left(\sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}\right)$

Theorem (Extended FTA) If $p(x)$ is an n^{th} degree polynomial with real or complex coefficients, then $p(x)$ factors completely into a product of form

$$p(x) = a(x-r_1)(x-r_2)\cdots(x-r_n) \quad \text{where } a, r_1, \dots, r_n \in \mathbb{C}.$$

Moreover, for every r_i with a non zero imaginary part, $\overline{r_i}$ is a r_j for some j.

Conjugation and norm

If $z = a+bi \in \mathbb{C}$ then $\bar{z} := a-bi$ (= complex conjugate of z)

and $\|z\| = \sqrt{a^2+b^2}$ (= norm of z).

Note: If $z \in \mathbb{R}$ then $\|z\| = \sqrt{a^2+0^2} = |a|$

For any $z \in \mathbb{C}$, $z\bar{z} = (a+bi)(a-bi) = a^2-(bi)^2 = a^2+b^2 = \|z\|^2$.

Define $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $z \in \mathbb{C}$. (Note: What does this mean?)

Thm This series converges for all $z \in \mathbb{C}$, and the corresponding function has all the usual properties of e^z

A Neat Consequence:

If $\theta \in \mathbb{R}$, $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i)^n \theta^n}{n!} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} \cdots$

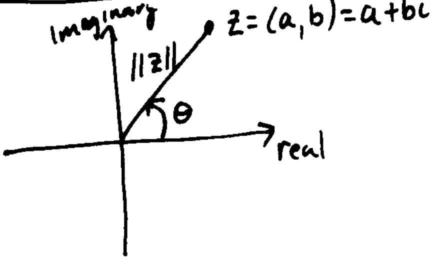
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) = \cos\theta + i\sin\theta$$

$\therefore \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ (or: $e^{(x+iy)} = e^x(\cos y + i\sin y)$)

Euler's Formula

In particular: $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i\cdot 0$, so: $\boxed{e^{i\pi} + 1 = 0}$

Polar Form / Argand diagram:

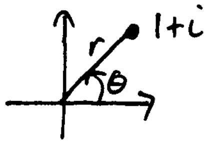


Note $z = \|z\|\cos\theta + i\|z\|\sin\theta$

$= \|z\| e^{i\theta}$ ← polar form

Examples

(1) $(1+i)^{27} = ?$ Put in polar form:



θ clearly $= \pi/4$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore (1+i) = (2^{1/2}) e^{i\pi/4}$

$\therefore (1+i)^{27} = 2^{27/2} e^{i27\pi/4} = 2^{27/2}\left(\cos\frac{27\pi}{4} + i\sin\frac{27\pi}{4}\right)$

$= 2^{27/2}\left(\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right) = 2^{13}(1-i)$

Note $z=(1+i)^{27}$ satisfies $z^{27} = 2^{13}\bar{z}$, so $z^{28} = 2^{13}\|z\|^2 = 2^{14}$.

$(1+i)$ is a 28^{th} root of 2^{14}, or a squareroot of ± 2.

(2) $\sqrt{1+i} = ?$ $(1+i)^{\frac{1}{2}} = (2^{1/2})^{\frac{1}{2}} e^{i\pi/8} = 2^{\frac{1}{4}}\left(\cos\pi/8 + i\sin\frac{\pi}{8}\right)$

This is *not* a unique answer: $(1+i)$ *also* equals $2^{\frac{1}{2}} e^{i9\pi/4}$,

so $(1+i)^{1/2} = 2^{1/4}\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$ which is the negation of our other answer.