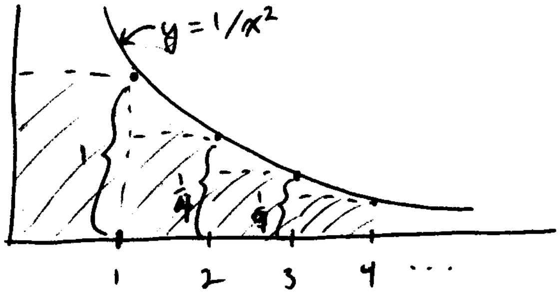
Integral test



EG $\sum_{n=1}^{\infty} \frac{1}{n^2}$

If $S_N = \sum_{n=1}^{N} \frac{1}{n^2}$,

Then S_N is monotone increasing, so $\lim_{N\to\infty} S_N$ is finite if & only if $\{S_N\}_{N=1}^{\infty}$ is bounded.

Note (picture) S_N = area of shaded region up to N

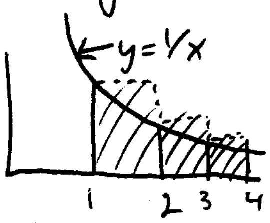
$$\leq 1 + \int_1^N \frac{1}{x^2}\,dx = 1 + \left(1 - \frac{1}{N}\right)$$

$$\leq 2$$

so $\{S_N\}$ bounded by 2

so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (in fact, ≤ 2)

EG $\sum_{n=1}^{\infty} \frac{1}{n}$ - try same thing (but know diverges!)



Then $S_N \geq \int_1^{N+1} \frac{1}{x}\,dx$, which $\nearrow \infty$ as $N \to \infty$ (p-test, p=1)

so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Conclusion: If we can squeeze a divergent improper integral under Σa_n, then Σa_n diverges

If we can squeeze Σa_n between 0 and a convergent improper integral, then Σa_n converges

(This requires $a_n \geq 0$ - why?)

Thm (Integral Test) Suppose $f: [1,\infty) \to (0,\infty)$ is continuous, decreasing (or nonincreasing); then:

$\sum f(n)$ converges if & only if $\int_1^{\infty} f(x)\,dx$ converges

$\sum f(n)$ diverges if & only if $\int_1^{\infty} f(x)\,dx$ diverges.

(Integral test continued)

EG $\sum \frac{1}{n^p}$ converges if $p>1$, diverges if $p \le 1$

EG $\sum n e^{-n}$: Let $f(x) = xe^{-x}$. f is continuous ✓

If f decreasing? $f'(x) = x\frac{d}{dx}(e^{-x}) + \left(\frac{d}{dx}x\right)e^{-x} = -xe^{-x} + e^{-x} = \underbrace{(e^{-x})}_{>0}\underbrace{(1-x)}_{<0 \text{ if } x>1}$

Which is <0 on $(1,\infty)$, so f decreasing on $[1,\infty)$

$$\int_1^\infty xe^{-x}\,dx = \lim_{R\to\infty}\int_1^R xe^{-x}\,dx \quad \boxed{\begin{array}{ll} u=x & dv=e^{-x}dx \\ du=dx & v=-e^{-x}\end{array}} \quad = \lim_{R\to\infty}\left[\left(-xe^{-x}\right|_1^R + \int_1^R e^{-x}\,dx\right]$$

$$= \lim_{R\to\infty}\left[e - \frac{R}{e^R} + e - e^{-R}\right] = 2e < \infty, \text{ so } \int_1^\infty xe^{-x}\,dx \text{ converges}$$

so $\sum n e^{-n}$ converges by Integral test

Remarks

(1) Alt. ways to show xe^{-x} decreasing (class)

(2) Not necessary to compute $\int_1^\infty xe^{-x}\,dx$, just show it converges.

WARNING: $\sum \frac{n+1}{n+100}$