MISCELLANEOUS SUBSTITUTIONS

IDEA: USE 'CLEVER' SUBSTITUTION TO ELIMINATE RADICALS ETC

EG ① $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}}$ $\boxed{\begin{array}{l}\text{Let } x=\tan\theta \\ dx=\sec^2\theta\, d\theta\end{array}}$ ← what if $x=1/u$ etc?

$$= \int \frac{\sec^2\theta\, d\theta}{(\tan\theta)\underbrace{\sqrt{\tan^2\theta+1}}_{\sec^2\theta}} = \int \frac{\sec\theta}{\tan\theta}\, d\theta$$

(Note $\theta=\tan^{-1}x$
$\Rightarrow -\pi/2 < x < \pi$
$\Rightarrow \sec\theta > 0$)

$$= \int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + c$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + C$$

② $\displaystyle\int \sqrt[3]{\frac{x+1}{x-1}}\ \frac{dx}{x+1}$

$\boxed{\begin{array}{l}\text{Let } u=\left(\dfrac{x+1}{x-1}\right)^{1/3} \quad \therefore x=\dfrac{u^3+1}{u^3-1} \\ dx=\dfrac{-6u^2}{(u^3-1)^2}\,du\end{array}}$

$$= \int u\left(\frac{1}{\left(\frac{2u^3}{u^3-1}\right)}\right)\frac{(-6u^2)}{(u^3-1)^2}\,du = -3\int\frac{du}{u^3-1} = -3\int\frac{du}{(u-1)(u^2+u+1)} \text{ etc}$$

③ $\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}$

Let x = something with both square and cube root, so $= u^6$

$\therefore dx = 6u^5\,du$

$$= \int \frac{6u^5\,du}{u^6(1+2u^3+u^2)}$$

note $2u^3+u^2+1 = (u+1)(2u^2-u+1)$

↑ guessed $u=-1$ a root, so $u+1$ divides

$$= \int \frac{6\,du}{u(u+1)(2u^2-u+1)}$$

$$\frac{6}{u(u+1)(2u^2-u+1)} = \frac{A}{u} + \frac{B}{u+1} + \frac{Cu+D}{2u^2-u+1}$$

$\therefore\ 6 = A(u+1)(2u^2-u+1) + Bu(2u^2-u+1) + (Cu+D)u(u+1)$

u^3: $0 = 3A + 2B + C$

$u=0$: $6 = A\cdot 1\cdot 1$ $\therefore$ $A=6$

$u=-1$: $6 = B(-1)(4)$ $\therefore$ $B = -3/2$

$\Big\}$ $C = -3A-2B = -18+3 = -15$, $C=-15$

$u=1$: $6 = 4A + 2B + (C+D)\cdot 2$

$= 24 - 3 + (-30) + 2D$ $\therefore$ $D = \frac{39-24}{2} = 15/2$

$\therefore$ Integral $= 6\int\frac{du}{u} + (-\frac{3}{2})\int\frac{du}{u+1}\ \ \frac{15}{2}\int\frac{-2u+1}{2u^2-u+1}\,du$

$= 6\ln|u| - \frac{3}{2}\ln|u+1| - \frac{15}{4}\int\frac{(4u-1)-1}{2u^2-u+1}\,du$

$= 6\ln|x|^{1/5} - \frac{3}{2}\ln|x^{1/5}+1| - \frac{15}{4}\ln|2u^2-u+1| + \frac{15}{4}\int\frac{du}{2u^2-u+1}$

$= 6\ln|x|^{1/5} - \frac{3}{2}\ln|x^{1/5}+1| - \frac{15}{4}\ln|\sqrt[3]{x}-\sqrt[6]{x}+1| + \frac{15}{2\sqrt{7}}\tan^{-1}\left(\frac{4x^{1/6}-1}{\sqrt{7}}\right) + C$

④ $$\int \frac{x^9\,dx}{\sqrt{1+x^2}} \qquad \text{Let } x=\tan\theta,\ dx=\sec^2\theta\, d\theta$$

$$= \int \frac{\tan^9\theta \sec^2\theta\, d\theta}{\sec\theta} = \int \tan^8\theta \sec\theta \tan\theta\, d\theta$$

$$= \int (\sec^2\theta - 1)^4 \sec\theta \tan\theta\, d\theta \qquad \boxed{\begin{array}{l} u=\sec\theta \\ du=\sec\theta\tan\theta \end{array}}$$

$$= \int (u^2-1)^4\, du \quad \underline{\text{etc}}$$

OR $$\boxed{\begin{array}{l}\text{Let } u=1+x^2,\ \text{so } x^2=u-1 \\ du=2x\,dx\end{array}}$$

$$= \frac{1}{2}\int \frac{x^8\, 2x\,dx}{\sqrt{1+x^2}} = \frac{1}{2}\int \frac{(u-1)^4\,du}{\sqrt{u}} = \frac{1}{2}\int \frac{u^4+4u^3+6u^2-4u+1}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int u^{7/2} - 4u^{5/2} + 6u^{3/2} - 4u^{1/2} + u^{-1/2}\,du$$

$$= \frac{1}{9}u^{9/2} - \frac{4}{7}u^{7/2} + \frac{6}{5}u^{5/2} - \frac{4}{3}u^{3/2} + u^{1/2} + C$$

$$= \frac{1}{9}(1+x^2)^{9/2} - \frac{4}{7}(1+x^2)^{7/2} + \frac{6}{5}(1+x^2)^{5/2} - \frac{4}{3}(1+x^2)^{3/2} + \sqrt{1+x^2} + C$$

⑤ $\int t^2\sqrt{t-5}\,dt$ $\boxed{\text{Let } u=t-5,\ du=dt}$

$= \int (u+5)^2\sqrt{u}\,du = \int (u^2+10u+25)\sqrt{u}\,du$

$= \int u^{5/2} + 10u^{3/2} + 25u^{1/2}\,du$ etc

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⑥ $\int \frac{\cos x\,dx}{\sin^3 x + \cos^3 x}$ $\begin{matrix}\div \cos^3 x \\ \div \cos^3 x\end{matrix}$ $= \int \frac{\sec^2 x\,dx}{\tan^3 x + 1}$ $\boxed{\begin{matrix}u=\tan x \\ du=\sec^2 x\,dx\end{matrix}}$

$= \int \frac{du}{u^3+1} = \int \frac{du}{(u+1)(u^2-u+1)} \underset{\substack{\uparrow \\ \text{PF} \\ \text{DECOMP}}}{=} \frac{1}{3}\int \frac{du}{u+1} - \frac{1}{3}\int \frac{(u-2)\,du}{u^2-u+1}$

etc

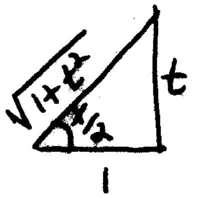
What substitution to use here?:

(a) $\int \frac{\ln(x+1)}{x+1}\,dx$ (b) $\int \frac{\sqrt{y}\,dy}{1+y\sqrt{y}}$ (c) $\int \frac{\sec^2\theta\,d\theta}{a+\tan^2\theta}$

(d) $\int r(2r+1)^{25}\,dr$ (e) $\int_{\pi/4}^{3\pi/4} e^{3\sin^2 x}\sin^{100} x\cos x\,dx$

$t = \tan\frac{x}{2}$ substitution $\left(\therefore x = 2\tan^{-1} t,\ dx = \frac{2}{1+t^2}\,dt\right)$

Turns <u>any</u> rational function of trig functions into a rational function.



$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,\frac{t}{\sqrt{1+t^2}}\,\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{2t}{1-t^2} \qquad \text{etc}$$

<u>EG</u>

$$\int \frac{\sin x\, dx}{\sin x + \cos^2 x} = \int \frac{\left(\frac{2t}{1+t^2}\right)\left(\frac{2}{1+t^2}\right) dt}{\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right)^2} \qquad \begin{matrix} \times (1+t^2)^2 \\ \times (1+t^2)^2 \end{matrix}$$

$$= \int \frac{4t\, dt}{2t(1+t^2) + (1-t^2)^2} = \int \frac{4t\, dt}{(2t^3+2t) + (t^4 - 2t^2 + 1)}$$

$$= \int \frac{4t\, dt}{t^4 + 2t^3 - 2t^2 + 2t + 1} \qquad \text{etc}$$