Def An <u>ordinary</u> <u>differential</u> <u>equation</u> (ODE) is an equation of form $F(x, y, y', y'', \dots, y^{(n)}) = 0$

(This is an <u>n^{th} degree</u> ODE)

A <u>solution</u> of an ODE is a function $y = f(x)$ rendering the ODE true

EG (1) $y = e^x$ is a sol'n of the eq'n $y' = y$. So is $y = Ce^x$ for constant C

(2) $y = \sin x$ solves $y'' + y = 0$. So does $y = \cos x$, or $y = A\sin x + B\cos x$.

(3) $y = \frac{\tan^{-1} x}{x}$ solves $xy' + y = \frac{1}{1+x^2}$ (Show!)

Def A linear ODE is an ODE that can be written:

$$P_n(x)\, y^{(n)} + P_{(n-1)}(x)\, y^{(n-1)} + \cdots + P_1(x)\, y' + P_0(x)\, y = Q(x)$$

where $Q(x)$, $P_i(x)$ are functions of x alone.

Note All examples above are linear ODEs. Note that (1) and (2) are linear differential equations with <u>constant</u> <u>coefficients</u>.

Goal find a solution/all solutions to a given ODE

<u>Simple example</u>

If the ODE is $y' = g(x)$ then $y = \int g(x)\,dx + C$ is a sol'n for <u>any</u> constant C. EG, all the functions $f(x) = \frac{x^3}{3} + C$ are sol'ns to the ODE $y' = x^2$; moreover, <u>all</u> sol'ns have this form.

<u>Initial Value Problems</u>

An IVP is an ODE together with one or more "initial conditions" (or "boundary conditions") that determine specific values for constants.

EG $\boxed{y' = x^2;\ y(0) = 3}$ has sol'n $f(x) = \frac{x^3}{3} + C$ where C must $= f(0) - \frac{0^3}{3} = 3$.

1^{st} order separable equations

Form: $y' = Q(x)R(y)$

To solve:

1. Divide both sides by $R(y)$: $\underbrace{A(y)\frac{dy}{dx} = Q(x)}$ $\quad (A(y) = 1/R(y))$
2. Integrate both sides $\quad$ or: $A(y)dy = Q(x)dx$
3. If appropriate, solve for y in terms of x

EG ① $y' = y$: $\frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx \Rightarrow \ln|y| + c = x + c'$

$\Rightarrow$ (combining constants) $\ln|y| = x + C \Rightarrow |y| = e^x e^c \Rightarrow y = ke^x$

② $xy' + y = y^2$: $x\frac{dy}{dx} = y^2 - y \Rightarrow \frac{dy}{y^2-y} = \frac{dx}{x} \Rightarrow \int \underbrace{\frac{dy}{y(y-1)}}_{\frac{1}{y-1} - \frac{1}{y}} = \int \frac{dx}{x}$

$\Rightarrow \ln|y-1| - \ln|y| = \ln|x| + c$

$\Rightarrow \left|\frac{y-1}{y}\right| = |x|e^c$, or $\frac{y-1}{y} = Cx$ (different C)

$\Rightarrow y = \frac{1}{1-Cx}$

Why this works:

If F is an antiderivative for A, $F'(y) = A(y)$, and G is an antiderivative for $Q(x)$, $G'(x) = Q(x)$, and $F(y) = G(x) + C$ implicitly defines y as a function of x, then we can differentiate implicitly:

$$\underbrace{F'(y)}_{A(y)}\frac{dy}{dx} = \underbrace{G'(x)}_{Q(x)} + 0 \quad \checkmark$$

Existence-Uniqueness Theorems

EG We know: $y = C_0 e^x$ is a sol'n to the IVP $\boxed{y' = y;\ y(0) = C_0}$

Can there be another?

Let $y = g(x)$ be a sol'n, i.e., $g'(x) = g(x)$ and $g(0) = C_0$.

To show: $g(x) = C_0 e^x$; equivalently, $g(x)e^{-x} = C_0$.

Note $\frac{d}{dx}\left(g(x)e^{-x}\right) = g'(x)e^{-x} + (-e^{-x})g(x) = g(x)e^{-x} - e^{-x}g(x) = 0$

$\therefore$ $g(x)e^{-x}$ is a constant function. Plug in $x=0$:

$g(0)e^0 = C_0 \cdot 1 \Rightarrow g(x)e^{-x} = C_0$ for all x $\therefore$ $\underline{g(x) = C_0 e^x}$ ✓

More generally:

Thm Consider the IVP $\boxed{y' = \varphi(x,y);\ y(a) = C_0}$ If

φ is "reasonably nice"* Then this IVP has one and only one solution on a neighborhood of $x = a$.

(* φ and $\frac{\partial \varphi}{\partial y}$ are continuous near (a, C_0)) ← Math 243

Prf Way way way beyond the scope of this course. We will prove some special cases later

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EG Let $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$, α not a positive integer.

Recall $f(x)$ is the series expansion for $(1+x)^\alpha$, and converges for $|x|<1$.

Note $y = (1+x)^\alpha$ satisfies the IVP $\boxed{(1+x)y' = \alpha y;\ y(0) = 1}$ (show!)

To show $f(x) = (1+x)^\alpha$, it suffices to show f satisfies the same IVP.

$$(1+x)f'(x) = (1+x)\sum_{n=1}^{\infty}\binom{\alpha}{n} n x^{n-1} = \sum_{n=1}^{\infty}\binom{\alpha}{n} n x^{n-1} + \sum_{n=1}^{\infty}\binom{\alpha}{n} n x^{n}$$

$$= \sum_{n=0}^{\infty}\binom{\alpha}{n+1}(n+1)x^n + \sum_{n=0}^{\infty}\binom{\alpha}{n} n x^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\left[\binom{\alpha}{n+1}(n+1) + \binom{\alpha}{n} n\right]}_{\text{Show That This } = \alpha\binom{\alpha}{n}\ :} x^n$$

$$\binom{\alpha}{n+1}(n+1) + \binom{\alpha}{n} n = \frac{\alpha(\alpha-1)\cdots(\alpha-(n+1)+1)}{(n+1)!}(n+1) + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} n$$

$$= \underbrace{\left[\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\right]}_{=\binom{\alpha}{n}} \left[\underbrace{\frac{(\alpha-(n+1)+1)\cancel{(n+1)}}{\cancel{n+1}}+n}_{=\alpha}\right]$$

✓

EG $\boxed{xy' + (1-x)y = e^{2x}}$

Equiv: $y' + \frac{1-x}{x}y = \frac{1}{x}e^{2x}$; put $I = e^{\int \frac{1-x}{x}dx} = e^{\ln x - x} = xe^{-x}$

$$\underbrace{xe^{-x}y' + (1-x)e^{-x}y}_{\frac{d}{dx}(xe^{-x}y)} = e^{x}$$

$\frac{d}{dx}(xe^{-x}y) = e^{x}$, $\therefore$ $xe^{-x}y = \int e^{x}dx = e^{x} + C$

$\therefore$ $y = \frac{e^{2x}}{x} + \frac{Ce^{x}}{x}$, any C

EG $\boxed{y'\sin x + y\cos x = 1}$ (on $(0,\infty)$)

$y' + y\cot x = \sec x$ (note assuming $\sin x \neq 0$, so interval $\not\ni k\pi$, $k \in \mathbb{Z}$)

$I = e^{\int \cot x\, dx} = e^{\ln|\sin x|} = \sin x$

Multiplying: $\underbrace{y'\sin x + y\cos x}_{\frac{d}{dx}(y\sin x)} = 1$ (! back where we started)

$\frac{d}{dx}(y\sin x) = 1$

$y\sin x = \int 1\,dx = x + C$

$y = \frac{x+C}{\sin x}$ (intervals where $\sin x \neq 0$)

Note If $C \neq 0$, $\lim_{x\to 0^+} y(x) = \pm\infty$, but if $C=0$, $\lim_{x\to 0^+} y(x) = 1$

Q: For what C is $\lim_{x\to\pi^-} y(x)$ finite?

2^{nd} order linear differential Equations with constant Coefficients

<u>Form</u>: $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = Q(x)$, a, b, c constants, $a \neq 0$.

<u>Today</u> Homogeneous case ($Q(x)=0$)

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0}$$

To solve: ① Form <u>auxiliary equation</u> $ar^2+br+c=0$

② Obtain roots of equation; 3 cases:

(a) Two distinct real roots $r_1 \neq r_2$. $\leftarrow ar^2+br+c = a(r-r_1)(r-r_2)$

Then sol'ns have form

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

(b) One repeated root, $r = r_1 = r_2$ $\leftarrow ar^2+br+c = a(r-r_1)^2$

Then sol'ns have form

$$\boxed{y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}}$$

(c) Two distinct <u>complex</u> roots, $r_1 = \alpha + \beta i$, $r_2 = \alpha - \beta i$

Then sol'ns have form:

$$\boxed{y = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)}$$

EG $\boxed{y''+y=0}$ Aux. eq'n: $r^2+1=0$, $r=0\pm 1i$

$\therefore$ Sol'n is $y=e^{0x}(c_1\cos x+c_2\sin x)=c_1\cos x+c_2\sin x$

EG $\boxed{y''-y=0}$ $\Rightarrow r^2-1=0$, $r=\pm 1$

$\therefore$ soln: $y=c_1e^{x}+c_2e^{-x}$

EG $\boxed{y''-6y'+9y=0}$ $\Rightarrow r^2-6r+9=0$, $(r-3)^2=0$ so $r=3$ is only root

$\therefore$ Sol'n is $y=c_1e^{3x}+c_2xe^{3x}$

Note

Suppose 2 complex roots, $\alpha\pm\beta i$. What if we plug into formula for 2 distinct real roots?

$$y=c_1e^{(\alpha+\beta i)x}+c_2e^{(\alpha-\beta i)x}=c_1e^{\alpha x}e^{i(\beta x)}+c_2e^{\alpha x}e^{i(-\beta x)}$$

$$=e^{\alpha x}\Big[c_1(\cos\beta x+i\sin\beta x)+c_2(\underbrace{\cos(-\beta x)}_{=\cos\beta x}+i\underbrace{\sin(-\beta x)}_{=-\sin\beta x})\Big]$$

$$=e^{\alpha x}\Big[\underbrace{(c_1+c_2)}_{A}\cos\beta x+\underbrace{i(c_1-c_2)}_{B}\sin\beta x\Big]$$

$$=e^{\alpha x}\big[A\cos\beta x+B\sin\beta x\big]$$

Note Given c_1, c_2 can find A, B

Given A, B can find c_1, c_2

$\therefore$ gives all sol'ns