Integration of rational functions

(A rational function is a ratio of two polynomials)

Some special cases:

1. $$\int \frac{dx}{(x-\alpha)^n} = \begin{cases} \ln|x-\alpha| + C & \text{if } n=1 \\ \dfrac{(x-\alpha)^{-n+1}}{-n+1} + C & \text{if } n\neq 1, n>0 \end{cases}$$

2. $\displaystyle\int \frac{dx}{x^2+ax+b}$, denominator irreducible (= doesn't factor):

- complete the square
- use suitable $\int$ formula or trig substitution

EG
$$\int \frac{dx}{x^2+3x+7} = \int \frac{dx}{(x+\frac{3}{2})^2+\frac{19}{4}} \quad \boxed{\begin{array}{l} u = x+3/2 \\ du = dx \end{array}} \quad = \int \frac{du}{u^2+\frac{19}{4}}$$

$$\boxed{\begin{array}{l} u = \frac{\sqrt{19}}{2}\tan\theta \\ du = \frac{\sqrt{19}}{2}\sec^2\theta\, d\theta \end{array}} \quad = \int \frac{\frac{\sqrt{19}}{2}\sec^2\theta\, d\theta}{\left(\frac{19}{4}\right)(1+\tan^2\theta)} = \frac{2}{\sqrt{19}}\theta + C$$

$$= \frac{2}{\sqrt{19}}\tan^{-1}\frac{2u}{\sqrt{19}} + C = \frac{2}{\sqrt{19}}\tan^{-1}\left[\frac{2}{\sqrt{19}}\left(x+\frac{3}{2}\right)\right] + C$$

-or-

$$\int \frac{dx}{(x+\frac{3}{2})^2+\frac{19}{4}} = \boxed{\begin{array}{l} x+3/2 = \frac{\sqrt{19}}{2}z \\ dx = \frac{\sqrt{19}}{2}dz \end{array}} \quad \int \frac{\frac{\sqrt{19}}{2}\,dz}{\frac{19}{4}z^2+\frac{19}{4}} = \frac{2}{\sqrt{19}}\int \frac{dz}{z^2+1}$$

$$= \frac{2}{\sqrt{19}}\tan^{-1}z + C = \frac{2}{\sqrt{19}}\tan^{-1}\left(\frac{2x+3}{\sqrt{19}}\right) + C$$

$\underline{3}$ $\displaystyle\int \frac{dx}{(x^2+ax+b)^n}$, $n>1$:

- complete square, use trig substitution

$\underline{EG}$ $\displaystyle\int \frac{dx}{(x^2+3x+7)^7} = \int \frac{dx}{\left[\left(x+\frac{3}{2}\right)^2+\frac{19}{4}\right]^7}$ $\boxed{\begin{array}{l} x+\frac{3}{2}=\frac{\sqrt{19}}{2}\tan\theta \\ dx=\frac{\sqrt{19}}{2}\sec^2\theta\, d\theta \end{array}}$

$$= \int \frac{\frac{\sqrt{19}}{2}\sec^2\theta\, d\theta}{\left[\frac{19}{4}(\tan^2\theta+1)\right]^7} = \left(\frac{4}{19}\right)^{13/2} \int \frac{d\theta}{\sec^{12}\theta}$$

$$= \left(\frac{4}{19}\right)^{13/2} \int \cos^{12}\theta\, d\theta \quad \text{(etc)}$$

$\underline{4}$ $\displaystyle\int \frac{x\,dx}{(x^2+ax+b)^n}$ $n \geq 1$, x^2+ax+b irreducible:

- modify numerator to be derivative of x^2+ax+b
- reduce to earlier cases

$\underline{EG}$ $\displaystyle\int \frac{x\,dx}{(x^2+3x+7)^7} = \frac{1}{2}\int \frac{2x+3-3}{(x^2+3x+7)^7}\,dx = \frac{1}{2}\int \frac{2x+3}{(x^2+3x+7)^7}\,dx - \frac{3}{2}\int \frac{dx}{(x^2+3x+7)^7}$

$\boxed{\begin{array}{l} u = x^2+3x+7 \\ du = (2x+3)\,dx \end{array}}$ $\displaystyle = \frac{1}{2}\underbrace{\int \frac{du}{u^7}}_{=\frac{u^{-6}}{-6}+C} - \frac{3}{2}\underbrace{\int \frac{dx}{(x^2+3x+7)^7}}_{\text{previous problem}}$

$= -\frac{1}{6}(x^2+3x+7)^{-6} + C$

$$\int \cos^{12}\theta \, d\theta \quad \text{using reduction formula:}$$

$$\boxed{\int \cos^n u \, du = \frac{1}{n}\cos^{n-1} u \sin u + \left(\frac{n-1}{n}\right)\int \cos^{n-2} u \, du}$$

$$\int \cos^{12}\theta \, d\theta = \frac{1}{12}\cos^{11}\theta \sin\theta + \frac{11}{12}\int \cos^{10}\theta \, d\theta$$

$$= \frac{1}{10}\cos^{9}\theta \sin\theta + \frac{9}{10}\int \cos^{8}\theta \, d\theta$$

$$= \frac{1}{8}\cos^{7}\theta \sin\theta + \frac{7}{8}\int \cos^{6}\theta \, d\theta$$

$$= \frac{1}{6}\cos^{5}\theta \sin\theta + \frac{5}{6}\int \cos^{4}\theta \, d\theta$$

$$= \frac{1}{4}\cos^{3}\theta \sin\theta + \frac{3}{4}\int \cos^{2}\theta \, d\theta$$

$$= \underbrace{\frac{1}{2}\cos\theta \sin\theta}_{=\frac{1}{4}\sin 2\theta} + \underbrace{\frac{1}{2}\int d\theta}_{\frac{\theta}{2}}$$

so $\int \cos^{12}\theta \, d\theta$

$$= \frac{1}{12}\cos^{11}\theta \sin\theta + \frac{11}{12}\left[\frac{1}{10}\cos^{9}\theta \sin\theta + \frac{9}{10}\left[\frac{1}{8}\cos^{7}\theta \sin\theta + \frac{7}{8}\left[\frac{1}{6}\cos^{5}\theta \sin\theta\right.\right.\right.$$

$$\left.\left.\left.\left. + \frac{5}{6}\left[\frac{1}{4}\cos^{3}\theta \sin\theta + \frac{3}{4}\left[\frac{1}{4}\sin 2\theta + \frac{\theta}{2}\right]\right]\right]\right]\right] + C$$

Some important facts about polynomials:

1 The degree of a polynomial is the highest power of the variable:

EG degree of $17x^9 - 3x^4 + x - 1$ is 9

$4 - x^4 + x^2$ is 4

$(-x^2 + 17x + 12)(9x^{40} + 17)$ is 42 (etc)

2 Division Algorithm: If $P(x)$ is a polynomial of degree m & $Q(x)$ is a polynomial of degree n

Then $P(x) = Q(x)A(x) + r(x)$, where $A(x), r(x)$ are polynomials and $0 \leq$ degree of $r(x) < n$

(In particular: If degree of $P(x) \geq$ degree of $Q(x)$,

Then $\frac{P(x)}{Q(x)} = A(x) + \frac{r(x)}{Q(x)}$, with deg. of $r <$ deg of Q)

3 Fundamental Theorem of Algebra

Any polynomial $P(x)$ can be fully factored into a product of:

(i) Linear terms (form $(x-r)$, where $P(r)=0$)

(ii) Irreducible quadratic terms (of form $x^2 + \alpha x + \beta$)

(iii) A constant

EG $13x^8 - 78x^7 - 221x^6 + 533x^5 + 1612x^4 + 5291x^3 + 3380x + 13026x^2 - 10920$

$= 13(x-7)(x+2)^3(x^2 - 5x + 3)(x^2 + 5)$