To integrate $\dfrac{P(x)}{Q(x)}$

<u>Step 1</u> Make sure $\deg(P) < \deg(Q)$

<u>EG</u> Given $\displaystyle\int \frac{4x^6-8x^5-5x^4+7x^3+8x^2+5x-9}{x^4-2x^3-x+2}\,dx$

$$
\begin{array}{r|l}
 & 4x^2-5 \\
\hline
x^4-2x^3-x+2 & 4x^6-8x^5-5x^4+7x^3+8x^2+5x-9 \\
 & 4x^6-8x^5-4x^3+8x^2 \\
\cline{2-2}
 & -5x^4+11x^3+\quad 5x-9 \\
 & -5x^4+10x^3+5x-10 \\
\cline{2-2}
 & x^3+1
\end{array}
$$

so integral

$$= \int 4x^2-5+\frac{x^3+1}{x^4-2x^3-x+2}\,dx \quad = \frac{4x^3}{3}-5x+\int\frac{x^3+1}{x^4-2x^3-x+2}\,dx$$

Step 2 Fully factor denominator:

x^4-2x^3-x+2 Note 1 is a root, so $(x-1)$ divides;

$$
\begin{array}{r|l}
 & x^3-x^2-x-2 \\
x-1 & x^4-2x^3-x+2 \\
 & x^4-x^3 \\ \hline
 & -x^3-x+2 \\
 & -x^3+x^2 \\ \hline
 & -x^2-x+2 \\
 & -x^2+x \\ \hline
 & -2x+2 \\
 & -2x+2 \\ \hline
 & 0
\end{array}
$$

Note 2 is a root of x^3-x^2-x-2, so $(x-2)$ divides:

$$
\begin{array}{r|l}
 & x^2+x+1 \\
x-2 & x^3-x^2-x-2 \\
 & x^3-2x^2 \\ \hline
 & x^2-x-2 \\
 & x^2-2x \\ \hline
 & x-2
\end{array}
$$

x^2+x+1 is irreducible, since quad. formula gives $\frac{-1\pm\sqrt{1-4}}{2}$ ← <0,

so $x^4-2x^3-x+2 = (x-1)(x-2)(x^2+x+1)$

Step 3 Set up partial fractions decomposition:

If $(x-r)^n$ appears in denominator, include terms:

$$\frac{A}{x-r} + \frac{B}{(x-r)^2} + \cdots + \frac{C}{(x-r)^n}$$

If $(x^2+ax+c)^n$ appears in denominator, include

$$\frac{A_1x+B_1}{(x^2+ax+b)} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \cdots + \frac{A_nx+B_n}{(x^2+ax+b)^n}$$

EG
$$\frac{x^3+1}{(x-1)(x-2)(x^2+x+1)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{Cx+D}{x^2+x+1}$$

Step 4 Cross-multiply to clear fractions, find unknown coefficients

EG
$$x^3+1 = A(x-2)(x^2+x+1) + B(x-1)(x^2+x+1) + (Cx+D)(x-1)(x-2)$$

Equate coefficients:

$$\left.\begin{aligned} x^3&: 1 = A+B+C \\ x^2&: 0 = -A-3C+D \\ x&: 0 = -A+2C-3D \\ 1&: 1 = -2A-B+2D \end{aligned}\right\} \overset{\text{eliminate } A}{\Rightarrow} \begin{aligned} 1&=B-2C+D \\ 1&=B+3C-3D \\ 3&=B+2C+2D \end{aligned} \quad \begin{aligned} &0=5C-4D \\ &2=-C+5D \end{aligned} \quad 10=21D$$

$\therefore D = \frac{10}{21}$, $C = 5D-2 = \frac{8}{21}$, $B = 1+2C-D = \frac{27}{21} = \frac{9}{7}$, $A = 1-B-C = -\frac{14}{21} = \frac{-2}{3}$

so
$$\frac{x^3+1}{(x-1)(x-2)(x^2+x+1)} = \frac{-2/3}{x-1} + \frac{9/7}{x-2} + \frac{8x+10}{21(x^2+x+1)}$$

Shortcut (not in book)

Can sometimes simplify by plugging values in for x:

$$x^3+1 = A(x-2)(x^2+x+1) + B(x-1)(x^2+x+1) + (Cx+D)(x-1)(x-2)$$

Try $x=0$: $1 = A(-2)(1) + B(-1)(1) + D(-1)(-2)$

$$\boxed{1 = -2A - B + 2D}$$

$x=2$: $9 = B(2-1)(4+2+1) = 7B \quad \therefore \boxed{B = \frac{9}{7}}$

$x=1$: $2 = A(-1)(3) + 0 + 0 \quad \therefore \boxed{A = -2/3}$

Need one more, no obvious #, so just pick one, say

$x=-1$: $0 = A(-3)(1) + B(-2)(1) + (-C+D)(-2)(-3)$

$$\boxed{0 = -3A - 2B - 6C + 6D}$$

... and solve as before.

Step 5 Use methods from ,

to integrate:

$$\int \frac{x^3+1}{x^4-2x^3-x+2}\,dx = -\frac{2}{3}\int \frac{dx}{x-1} + \frac{9}{7}\int \frac{dx}{x-2} + \int \frac{8x+10}{21(x^2+x+1)}\,dx$$

$$= -\frac{2}{3}\ln|x-1| + \frac{9}{7}\ln|x-2| + \frac{1}{21}\left[4\int \frac{2x+1+\frac{10}{4}-1}{x^2+x+1}\,dx\right]$$

$$= -\frac{2}{3}\ln|x-1| + \frac{9}{7}\ln|x-2| + \frac{4}{21}\int \frac{2x+1}{x^2+x+1}\,dx + 2\int \frac{dx}{\underbrace{x^2+x+1}_{(x+\frac{1}{2})^2+\frac{3}{4}}}$$

$$\boxed{\begin{aligned} x+\tfrac{1}{2} &= \tfrac{\sqrt{3}}{2}\tan\theta \\ dx &= \tfrac{\sqrt{3}}{2}\sec^2\theta\, d\theta \end{aligned}}$$

$$= -\frac{2}{3}\ln|x-1| + \frac{9}{7}\ln|x-2| + \frac{4}{21}\ln(x^2+x+1)$$

$$+ 2\cdot \int \frac{\frac{\sqrt{3}}{2}\cancel{\sec^2\theta}\, d\theta}{\frac{3}{4}\cancel{\sec^2\theta}}$$

$$= \frac{4\sqrt{3}}{3}\theta = \frac{4\sqrt{3}}{3}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

ANOTHER EXAMPLE:

$$\int \frac{x^6+4x^4-x^3+2x^2-x-2}{x^5-x^4+4x^3-4x^2+4x-4}\,dx = \int x+1+\frac{x^4-x^3+2x^2-x+2}{(x-1)(x^2+2)^2}\,dx$$

PF decom:

$$\frac{x^4-x^3+2x^2-x+2}{(x-1)(x^2+2)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

$$x^4-x^3+2x^2-x+2 = A(x^2+2)^2 + (Bx+C)(x-1)(x^2+2) + (Dx+E)(x-1)$$

$x=1$: $3=9A \quad \therefore \boxed{A=1/3}$

$x=0$: $\boxed{2=4A-2C-E} \quad \therefore$

Coeff of x^4: $\boxed{1=A+B} \quad \therefore \boxed{B=1-A=2/3}$

Coeff of x^3: $\boxed{-1=\quad C-B} \quad \therefore \boxed{C=B-1=-1/3}$

$\therefore E = 4A-2C-2 = 4/3+2/3-2 = 0$

$x=-1$: $3 = 9A+(C-B)(-2)(3)+(E-D)(-2) = 3+(-1)(-2)(3)+(-D)(-2)$

$\therefore \boxed{D=-1}$

$$\therefore \text{Integral} = \frac{x^2}{2}+x+\frac{1}{3}\int\frac{dx}{x-1}+\int\frac{\frac{2}{3}x-\frac{1}{3}}{x^2+2}\,dx - \int\frac{x\,dx}{(x^2+2)^2}$$

$$= \frac{x^2}{2}+x+\frac{1}{3}\ln|x-1| + \frac{1}{3}\int\frac{2x\,dx}{x^2+2} - \frac{1}{3}\int\frac{dx}{x^2+2} - \frac{1}{2}\int\frac{2x\,dx}{(x^2+2)^2}$$

$$= \frac{x^2}{2}+x+\frac{1}{3}\ln|x-1| + \frac{1}{3}\ln(x^2+2) - \frac{1}{3}\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - \left(\frac{1}{2}\right)\frac{1}{x^2+2} + C$$