Infinite Series and Sequences: Some Tips and Traps

Math 242 Spring 2008 (Prof. Ross's Lecture)

1. Make sure you understand the difference between convergence of a **sequence**, and convergence of a **series**. If a series $\sum a_n$ converges, must the sequence a_n converge? If so, to what? If not, what is a counterexample? Similarly, if the sequence a_n converges, must the series $\sum a_n$ converge? Does your answer change if you specify that the sequence a_n converges to 0? Why or why not?

   Confusing convergence of a sequence and convergence of a series is the most common mistake made by Calc. II students, and often leads to the loss of many points in exams. If you are asked whether a series $\sum a_n$ converges, and after doing some manipulations of a_n you say that "it converges", you might get no points! Reason: I won't know whether by "it" you mean the series or the sequence.

2. When you show that a series converges or diverges, **make sure you name the test that you use.**

3. If you are using a procedure that involves inequalities, such as the comparison test or integral test, or an estimate involving remainders, **pay close attention to the direction of your inequalities**. The statements "$a < b$" and '$a > b$" are very different! You should *understand* (not just memorize) why the inequalities in the comparison test (etc) go the way they do.

4. (TIP) When possible, use order of magnitude arguments (such as the convergence hierarchy we had in lecture) to get a sense of how a sequence or series behaves, and to help you choose a test before proving convergence or divergence. The hierarchy results can often also be used in place of a L'Hôpital's Rule computation.

   For example, to determine the convergence of $\sum \frac{1}{\ln(n)}$ it is much easier to compare to $\sum \frac{1}{n}$ than to use an integral test.

5. (TIP) When verifying that a sequence or function is decreasing, showing that the derivative of a corresponding function is < 0 is sometimes the only way, but if there are alternatives they are most likely easier. Give some thought to the form of the problem before rushing blindly ahead!

   For example, suppose to show that the sequence $a_n = \frac{n+5}{n\ln(n)-7}$ is decreasing, if you divide numerator and denominator by n you get $a_n = \frac{1+5/n}{\ln(n)-7/n}$. Now, the numerator is obviously decreasing (as n increases, $5/n$ decreases, so so must $1 + 5/n$), and the denominator is inreasing ($\ln(n)$ is certainly increasing, and since $7/n$ is decreasing its negation $-7/n$ is increasing, and the sum of two increasing functions is clearly increasing), so the quotient must be decreasing.

   Note, in this example, that even if you're not comfortable with this kind of reasoning, dividing by n was still useful. If you want to show that a function $f(x)$ has negative derivative, it a little easier to work with the derivative of $\frac{1+5/x}{\ln(x)-7/x}$ than it is with $\frac{x+5}{x\ln(x)-7}$ (try them)!

   Here's another example to try both ways: $a_n = \frac{\sqrt{n}}{n+\sqrt{n}\ln(n)}$

6. (TRAP) We've seen lots of examples where some limit had to satisfy some conditions to make a test work. These conditions vary from test to test. For example, being "< 1" is the important condition for the root and ratio tests, being "$= 0$" for the 'bad' test (aka divergence test or alternating series test), being "finite and nonzero" for the limit comparison test, etc. (Of course, in each of these examples the thing we are taking the limit of is different!) Make sure you are clear on which conditions go with which test. You might find it a useful review exercise to make a table with test name, thing you need to take a limit of, what conditions on the limit you are seeking, and when this test is useful.

7. (TIP) When looking for a pattern in a sequence or series, writing out several terms will help you see the pattern. Don't simplify when you do this - it is often easier to spot the pattern if you leave terms as products, sums, etc. (This will be especially true for Taylor series, coming up after the exam.)

8. (TRAP) Even if the form of a series or sequence looks easy, write out a few terms anyway to avoid gotchas. For example, the series $\sum(-1)^n \frac{cos(n\pi)}{n}$ LOOKS like an alternating series (which might converge if true since $\frac{|cos(anything)|}{n} \to 0$ as $n \to \infty$), but in fact it is not (write out the first 3-4 terms).

9. (BIG TIP) Work lots of problems! Many problems are quite easy once you learn to recognize the form, and if you have worked several examples of each kind then many of the exam problems will be very easy. On the other hand, if you have to take the time to treat each problem as something completely new, the exam will be long and painful.