Continuous Functions

1 Definitions

Definition 1.1 *The function f is*

- continuous *at c provided* $\lim_{x \to c} f(x) = f(c)$
- left-continuous *at c provided* $\lim_{x \to c^-} f(x) = f(c)$
- right-continuous *at c provided* $\lim_{x \to c^+} f(x) = f(c)$

For these definitions to make sense we need to say where f is defined:

For continuity f needs to be defined in some "neighborhood" of c, that is, on some interval (a, b) with $a < c < b$. (Of course, f could be defined on a *bigger* set, possibly all real numbers, it just needs to be defined *at least* on a neighborhood of c.)

For left-continuity f needs to be defined in some neighborhood to the left of c, that is, on some interval $(a, c]$ (and maybe more).

For right-continuity f needs to be defined in some neighborhood to the right of c, that is, on some interval $[c, b)$ (and maybe more).

Now, we extend "continuity at a point" to "continuity on an interval":

Definition 1.2 *The function f is* continuous on the interval I *provided f is continuous at every point of I. (If I includes one or both endpoints then this is interpreted as left-continuity or right-continuity, as appropriate.)*

The intuitive meaning of continuity is:

- f is continuous if you can compute its limit by plugging in.
- f is continuous if you can graph it without lifting your pencil from the paper.
- $y = f(x)$ is continuous if tiny changes in x produce tiny changes in y

2 Properties

Properties of limits immediately become properties of continuous functions:

Theorem 2.1

1. *Polynomials are continuous everywhere.*
2. *Rational functions, that is functions of the form $f(x) = \frac{p(x)}{q(x)}$ with $p(x), q(x)$ polynomials, are continuous wherever defined (so f is continuous wherever $q(x) \neq 0$).*
3. *$|x|$ is continuous for all x*
4. *$\sqrt{x}$ is continuous for all $x \geq 0$*
5. *$\lim\limits_{x\to c} \sin x = \sin c$, so $\sin x$ is continuous for all x. Similarly for $\cos x$.*

(Proof in class for $\sin x$ when $x \neq 0$. The rest we've done already.)

Theorem 2.2 *Suppose f, g are continuous at $x = c$ (or on I). Then:*

1. *$f(x) + g(x), f(x) - g(x), f(x)g(x)$ are continuous at $x = c$ (or on I).*
2. *$\frac{f(x)}{g(x)}$ is continuous at $x = c$ provided $g(c) \neq 0$.*

(The proof follows from the corresponding properties for limits. **Note that all these results hold for one-sided continuity as well.**)

Cor 2.1 *$\tan x, \cot x, \sec x, \csc x$ are all continuous wherever defined.*

Theorem 2.3 *Suppose* $\lim_{x\to c} g(x) = L$ *and* f *is defined in a neighborhood of* L *and is continuous at* L*. Then* $\lim_{x\to c} f(g(x)) = f(L)$.

(Proof: Later.)

Examples:

1. $\lim_{x\to 2} \sqrt{x + \sin(x^2)} = ?$

2. Where is the function $\frac{x^3 - 37x}{\sqrt{\sin(2x)}}$ continuous?

3. Where is the function $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ continuous?

4. Redefine the function $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 2 & \text{if } x = 0 \end{cases}$ at one point so as to make it continuous everywhere.

5. Find all values of α, β so that the function $f(x) = \begin{cases} 1 & \text{if } x > 3, \\ \alpha x - 10 & \text{if } x < 3 \\ \beta & \text{otherwise} \end{cases}$ is continuous everywhere.