Introduction to vectors

Notation: \( \mathbb{R} \) the real numbers

\( \mathbb{R}^2 \) = the set \( \{(a, b) : a, b \in \mathbb{R}\} \) of ordered pairs of real numbers

= “the (Euclidean) plane,” or “Euclidean 2-space” (relative to a coordinate system)

\( \mathbb{R}^3 \) = the set \( \{(a, b, c) : a, b, c \in \mathbb{R}\} \) of ordered triples of real numbers

= “Euclidean 3-dimensional space” (relative to a coordinate system)

\( \mathbb{R}^n \) = the set \( \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}\} \) of ordered \( n \)-tuples of real numbers

= “Euclidean \( n \)-dimensional space” (relative to a coordinate system)
Goal: Establish mathematical object representing displacements in $\mathbb{R}^2, \mathbb{R}^3$, or even $\mathbb{R}^n$

**Definition; Text:** A vector in the plane is a pair 
$\langle a, b \rangle \in \mathbb{R}^2$

Similarly, a vector in 3-space is a triple $\langle a, b, c \rangle \in \mathbb{R}^3$

More generally, an *n-dimensional vector* is an n-tuple 
$\langle u_1, u_2, \ldots, u_n \rangle \in \mathbb{R}^n$

Note: we use the $\langle a, b \rangle$ notation instead of $(a, b)$ to emphasize the role of the tuple.

**Components:** The values $u_1, u_2, \ldots$ are called the components of $\langle u_1, u_2, \ldots, u_n \rangle$

**Warning:** A vector, as a conceptual object, should exist independently of coordinate system. (Picture)

To say that the vector is this n-tuple is therefore not quite correct.

A vector is an ‘ideal representation’ of a displacement in the plane (space, etc.), which has *magnitude* and *direction*.

It is more honest to say:

We *assume* some coordinate system is fixed, and then ‘identify’ the space of n-dimensional vectors with $\mathbb{R}^n$

Can describe vectors pictorially.
Operations.
Let \( \mathbf{u} = \langle u_1, u_2 \rangle, \mathbf{v} = \langle v_1, v_2 \rangle \), and \( c \in \mathbb{R} \) (a scalar)

**Vector addition:** \( \mathbf{u} + \mathbf{v} := \langle u_1 + v_1, u_2 + v_2 \rangle \)

**Scalar multiplication:** \( c \mathbf{u} := \langle cu_1, cu_2 \rangle \)

Examples (class)
These are the two operations that characterize an abstract vector space (Math 411)

Other operations can be defined in terms of these, eg:
\(-\mathbf{u} := (-1)\mathbf{u}\)
\(\mathbf{u} - \mathbf{v} := \mathbf{u} + (-\mathbf{v})\)

More generally we can define the same operations for \( \mathbb{R}^n, \quad n \geq 3 \):
Let \( \mathbf{u} = \langle u_1, u_2, \cdots, u_n \rangle, \mathbf{v} = \langle v_1, v_2, \cdots, v_n \rangle \)

**Vector addition:** \( \mathbf{u} + \mathbf{v} := \langle u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n \rangle \)

**Scalar multiplication:** \( c \mathbf{u} := \langle cu_1, cu_2, \cdots, cu_n \rangle \)
Other operations, and special vectors.

**Equality:** \( \langle u_1, u_2, \ldots, u_n \rangle = \langle v_1, v_2, \ldots, v_n \rangle \) provided \( u_i = v_i \) for all \( i \).

**Zero vector:** \( \mathbf{0} := \langle 0, 0, \ldots, 0 \rangle \). Note that there is a different zero vector for each \( \mathbb{R}^n \).

**Standard component vectors:** (In \( \mathbb{R}^2 \)) \( \mathbf{i} := \langle 1, 0 \rangle, \mathbf{j} := \langle 0, 1 \rangle \).

(In \( \mathbb{R}^3 \)) \( \mathbf{i} := \langle 1, 0, 0 \rangle, \mathbf{j} := \langle 0, 1, 0 \rangle, \mathbf{k} := \langle 0, 0, 1 \rangle \).

**Directions:** \( \mathbf{u} \) and \( \mathbf{v} \) have the same direction if \( \mathbf{u} = c \mathbf{v} \)
for some \( c > 0 \)

\( \mathbf{u} \) and \( \mathbf{v} \) have opposite directions if \( \mathbf{u} = c \mathbf{v} \) for some
\( c < 0 \)

\( \mathbf{u} \) and \( \mathbf{v} \) are parallel if \( \mathbf{u} = c \mathbf{v} \) or \( \mathbf{v} = c \mathbf{u} \) for some
\( c \in \mathbb{R} \)

**Length:** \( |\langle u_1, u_2, \ldots, u_n \rangle| = \text{the length or norm of } \langle u_1, u_2, \ldots, u_n \rangle \)
\[ = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \]

(Sometimes you'll see \( \| \mathbf{u} \| \) instead of \( | \mathbf{u} | \).)

**Examples (class)**

\[ \| \mathbf{u} \| \]

\[ \| \mathbf{u} \| = \sqrt{9+9+9+9} = 3 \sqrt{3} \]

\[ \mathbf{p} = \langle 0, 1, 1 \rangle \]

\[ \mathbf{q} = \langle 3, 4, -1 \rangle \]

\[ \mathbf{m} = \mathbf{p} \mathbf{q} = \langle 3-0, 4-1, -1-2 \rangle = \langle 3, 3, -3 \rangle \]

\[ \| \mathbf{m} \| = \sqrt{9+9+9} = 3 \sqrt{3} \]
Basic algebraic properties of vector operations

**Theorem:** Vector addition is commutative and associative,
\[ u + v = v + u \]
\[ (u + v) + w = u + (v + w) \]

Multiplication by scalars is associative,
\[ c(du) = (cd)u \]
and satisfies the two distributive laws
\[ c(u + v) = cu + cv \]
\[ (c + d)u = cu + du \]

(Here \( c, d \) are scalars, \( u, v \) are vectors.)