Some Applications of the theory; Approximations, Properties of $\mathbb{R}$ and the Intermediate Value Theorem

1 Approximations

Idea:

To find $f(b)$ if it is easy to compute $f(a)$ and $f'(a)$, where $a \simeq b$:

$$\begin{aligned} f(b) &= f(a) + f'(a)(b-a) + (\epsilon)(b-a) \qquad \text{(where } \epsilon \to 0 \text{ as } b \to a\text{)} \\ &\simeq f(a) + f'(a)(b-a) \qquad \text{(since } (\epsilon)(b-a) \text{ is small even compared to } (b-a)\text{)} \end{aligned}$$

Example Suppose we want to approximate 2.07^8.

Put $f(x) = x^8$, and note that $f'(x) = 8x^7$. Then (with $a = 2, \quad b = 2.07$):

$$\begin{aligned} 2.07^8 &= f(2.07) \\ &= f(2) + f'(2)(2.07 - 2) + \text{small error} \\ &\simeq f(2) + f'(2)(2.07 - 2) \\ &= 2^8 + 8(2^7)(.07) \\ &= 256 + 8(128)(.07) \\ &= 327.68 \end{aligned}$$

Exercises There are some exercises in the book appropriate to this application, although the text puts them in a section about *differentials*. You should look at:

§3.11 #19 − 30 (using the above method instead of differentials)

2 LUB and GLB

▷What is the difference between $\mathbb{R}$ and $\mathbb{Q}$?◁

- $\sqrt{2}$ - a "gap" in $\mathbb{Q}$
- How do we make this "gap" notion (a) more general, and (b) precise?

Definition 2.1 *Suppose A is a subset of $\mathbb{R}$.*

- *A is* bounded from above *if there is an $M \in \mathbb{R}$ with $x \leq M$ for every $x \in A$*
- *A is* bounded from below *if there is an $M \in \mathbb{R}$ with $x \geq M$ for every $x \in A$*

(Pictures - class)

AXIOM: If A is a subset of $\mathbb{R}$ which is bounded above, then A has a *least upper bound*(LUB); that is, there is an upper bound α such that α is $\leq$ every other upper bound for A.

(Similarly, every set which is bounded from below has a *greatest lower bound*, a GLB.)

(Pictures - class.)

For example, $\{x \in \mathbb{Q} : x^2 < 2\}$ is a subset of $\mathbb{Q}$ (hence of $\mathbb{R}$) which is bounded above, so it has a least upper bound. That LUB is $\sqrt{2}$. Note $\mathbb{Q}$ doesn't have such a thing.

3 IVT and Bolzano

The first useful application of this is a pair of results about continuous functions. (One is really just a special case of the other.)

They verifies the intuitive idea that continuous functions have no breaks or jumps.

Theorem 3.1 *(Bolzano) Suppose f is a continuous function on the closed interval $[a, b]$, and that $f(a) < 0 < f(b)$. Then $f(c) = 0$ for some $c \in (a, b)$*

(picture; proof in class)

Applications:

1. $f(x) = x^2 + 5x - 7$ has a root (in fact, at least two)
2. Every polynomial with odd degree has a root
3. (Important!) Sign-preservation of nonzero continuous functions: If a continuous $f \neq 0$ on (a, b) then f never changes sign on (a, b)
4. If $f : [a, b] \to \mathbb{R}$ is continuous, $f(a) < a, f(b) > b$. Show that f has a *fixed point* (that is, there is a c with $f(c) = c$).

Theorem 3.2 *(Intermediate Value Theorem) Suppose f is a continuous function on the closed interval $[a, b]$, and that $f(a) < d < f(b)$. Then $f(c) = d$ for some $c \in (a, b)$*

(picture; proof in class)