

Concavity, and applications

1 Review

Extreme Value Theorem A continuous function on a closed interval attains a min and max

Fermat If f has a local min or max at a point c then either f is not differentiable at c or $f'(c) = 0$

(Cor) To optimize (=find absolute min and/or max of) a continuous function on a closed interval, it suffices to look at its values at critical points and endpoints, and pick the one here f is smallest/largest.

Remark This list also contains all local minima and maxima, but no obvious way to identify which is which.

Proposition: Suppose f is continuous on a neighborhood of $x = c$. If f is increasing on the left of c , and decreasing on the right, then f has a local max at $x = c$. If f is decreasing on the left of c , and increasing on the right, then f has a local min at $x = c$.

(Cor. to MVT)

- If $f' > 0$ on the interval (a, b) then f is increasing on (a, b)
- If $f' < 0$ on the interval (a, b) then f is decreasing on (a, b)
- If $f' = 0$ on the interval (a, b) then f is constant on (a, b)

Remark: The last result is true on $[a, b]$ provided f is continuous on this interval

1st Derivative Test: Suppose f is differentiable on a neighborhood of $x = c$. If $f' > 0$ on the left of c , and $f' < 0$ on the right, then f has a local max at $x = c$. If $f' < 0$ on the left of c , and $f' > 0$ on the right, then f has a local min at $x = c$.

Examples:

- $f(x) = \sin^2(x) + \sqrt{2}\sin(x)$; where is f increasing/decreasing?
- $f(x) = |x - 1| + \frac{1}{1+|x|}$; find local and global minima/maxima
- $f(x) = \begin{cases} x^3; & x < 1 \\ 0; & x = 1 \\ x^2 - 3x + 3; & x > 1 \end{cases}$; find min/max of f on $[-1, 2]$ (if such exists)

2 Concavity

Suppose $a < c < b$ and that f' exists throughout the interval (a, b) .

Concavity: f is **concave up** if f' is nondecreasing in (a, b)

f is **concave down** if f' is nonincreasing in (a, b)

- Remarks:**
1. Compare to text
 2. “concave up” = ”convex”; “concave down” = ”concave”
 3. Concavity for functions not differentiable everywhere

Note: If $f'' > 0$ on an interval then f' is increasing on this interval, so f is concave up on the interval.

If $f'' < 0$ on an interval then f' is decreasing on this interval, so f is concave down on the interval

2nd Derivative Test: Suppose f'' exists in a neighborhood of a point c , and $f'(c) = 0$

1. If $f''(c) > 0$ then f has a local **minimum** at c
2. If $f''(c) < 0$ then f has a local **maximum** at c
3. If $f''(c) = 0$ then the test gives **no information**

Examples:

1. $f(x) = x^2$ (at $x = 0$)
2. $f(x) = -x^2$ (at $x = 0$)
3. $f(x) = x^3$ (at $x = 0$)
4. $f(x) = (1 - x)(1 + x)^3$ (recall: $f'(x) = 2(1 + x)^2(1 - 2x)$)