Continuous Functions

Theorem 0.1. *(Continuity of basic functions)*

1. $f(x) = k$ *is continuous for all* x*, where* k *is a constant.*
2. *Any polynomial* $P(x)$ *is continuous for all* x
3. *If* $f(x)$ *is a rational function, i.e.* $f = \frac{P(x)}{Q(x)}$ *where* $P(x)$ *and* $Q(x)$ *are polynomials, then* f *is continuous at every* x *for which* $Q(x) \neq 0$
4. $f(x) = \sqrt{x}$ *is continuous at all* $x > 0$*, and is right-continuous at* 0
5. $\sin(x)$ *and* $\cos(x)$ *are continuous for all* x
6. $\tan(x)$ *and* $\sec(x)$ *are continuous for all* $x \neq k\pi/2, k$ *an odd integer.*
7. $\cot(x)$ *and* $\csc(x)$ *are continuous for all* $x \neq k\pi, k$ *an integer.*

Proof. These all follow from the comparable properties for limits. For example, for any c, $\lim_{x \to c} \sin(x) = \sin(c)$, so $\sin(x)$ is continuous at such a c. □

Theorem 0.2. *(Construction rules)*

1. *If $f(x)$ is continuous at c and k is a constant, then $kf(x)$ is continuous at c.*

   If $f(x)$ is continuous on an interval I and k is a constant, then $kf(x)$ is continuous on I.

2. *If f and g are continuous at c, then so are $f+g$, $f-g$, and fg.*

   If f and g are continuous on an interval I, then so are $f+g$, $f-g$, and fg.

3. *If f and g are continuous at c, and $g(c) \neq 0$, then $\frac{f(x)}{g(x)}$ is continuous at c.*

   If f and g are continuous on an interval I, and $g(x) \neq 0$ for all $x \in I$, then $\frac{f(x)}{g(x)}$ is continuous on I.

Proof. These all follow from the comparable properties for limits. For example, if f and g are continuous at c, then

$$\lim_{x \to c}(f+g)(x) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)$$

□

Theorem 0.3. *(a) If f is continuous at c and* $\lim_{x \to a} g(x) = c$ *then* $\lim_{x \to a} f \circ g(x) = f(c)$.

(b) If g is continuous at a and f is continuous at $g(a)$ then $f \circ g$ is continuous at a.

Proof. See text for (a). (b) is an immediate consequence of (a). □

Examples: Class