

## Cross Product

1. Overview
2. Definition of *determinants* in  $\mathbb{R}^2$  and  $\mathbb{R}^3$
3. Algebraic definition of cross-product:

$$\text{If } \mathbf{a} = \langle a_1, a_2, a_3 \rangle, \mathbf{b} = \langle b_1, b_2, b_3 \rangle,$$

$$\mathbf{a} \times \mathbf{b} := \langle a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1 \rangle$$

4. Easier definition with determinants

## Algebraic properties of the cross product

**Theorem:** For all vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  in  $\mathbb{R}^3$  and  $c \in \mathbb{R}$ :

- (a)  $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$  (skew symmetry)
- (b)  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$  (distributive law)
- (c)  $c(\mathbf{a} \times \mathbf{b}) = (c\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (c\mathbf{b})$
- (d)  $\mathbf{a} \times (c\mathbf{a}) = \mathbf{0}$
- (e)  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$  (scalar triple product)
- (f)  $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$  (orthogonality  $\mathbf{a}$ )
- (g)  $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$  (orthogonality to  $\mathbf{b}$ )

**Proof:** From the definitions

**Examples Class**

## Geometric properties of the cross product

**Theorem** For all vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  in  $\mathbb{R}^3$ :

- (a) (Important!)  $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin \theta$   
(where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ )
- (b)  $\mathbf{a} \times \mathbf{b} = \mathbf{0}$  if and only if  $\mathbf{a} \parallel \mathbf{b}$
- (c)  $\mathbf{a} \times \mathbf{b} = (\|\mathbf{a}\|\|\mathbf{b}\| \sin \theta)\mathbf{n}$   
(where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$  and  $\mathbf{n}$  is a unit vector  $\perp \mathbf{a}, \mathbf{b}$ )
- (d)  $\|\mathbf{a} \times \mathbf{b}\| =$  area of the parallelogram with edges  $\mathbf{a}$  and  $\mathbf{b}$
- (e)  $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| =$  volume of the parallelepiped with edges  $\mathbf{a}, \mathbf{b}$ , and  $\mathbf{c}$

**Proof:** Class

**Examples** Class