Vectors (cont)

Definition: Dot Product (or scalar product):

Let $\mathbf{u} = \langle u_1, u_2, \cdots, u_n \rangle$, $\mathbf{v} = \langle v_1, v_2, \cdots, v_n \rangle$

$\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$

Remark: Odd kind of multiplication

Examples: (Class)

Important note:

$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$$

Algebraic properties of the dot product

Theorem: The dot product is commutative,

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

distributes with vector addition,

$$(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} + \mathbf{v})$$

associates with scalar mutiplication,

$$c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$$

Moreover,

$$\mathbf{u} \cdot \mathbf{u} \geq 0$$

with equality if and only if $\mathbf{u}$ is the zero vector

Proof: text and class

Note: Which operations make sense?

Since $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$, some of the above properties can be written as **basic norm properties**:

Theorem: Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$, $c \in \mathbb{R}$ a scalar. Then:

$\|\mathbf{u}\|^2 > 0$ if $\mathbf{u} \neq \mathbf{0}$, and $\|\mathbf{0}\|^2 = 0$

$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

$\|\mathbf{u}\| > 0$ if $\mathbf{u} \neq \mathbf{0}$, and $\|\mathbf{0}\| = 0$

$\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$

$\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\| = 1$ if $\mathbf{u} \neq \mathbf{0}$ (normalization of $\mathbf{u}$)

Proof: Class

Examples Class

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, can interpret them as geometric rays with a common base point (picture; class)

There is a well-defined angle θ between them, $0 \leq \theta \leq \pi$.

This extends to $\mathbb{R}^n$.

Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, and θ is the angle between them, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

Proof: Class; text

Some geometric properties of dot product:

Corollary: Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in R^n, θ the angle between them. Then:

$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$

$\mathbf{u} \perp \mathbf{v}$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ (Cauchy-Schwarz Inequality – not in text)

$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (Triangle Inequality – not in text)

Proofs: Text, class

Direction Angles/Cosines: If $\mathbf{u} =$ is a vector in $\mathbb{R}^2$, the *direction angles* are the angles which $\mathbf{u}$ makes with $\mathbf{i}$ and $\mathbf{j}$ (that is, with the x- and y-axes), and the *direction cosines* are the cosines of these angles.

Similarly, if $\mathbf{u}$ is a vector in $\mathbb{R}^3$, the *direction angles* are the angles which $\mathbf{u}$ makes with **i, j,** and **k** (that is, with the x-, y-, and z-axes), and the *direction cosines* are the cosines of these angles.

Theorem: If $\mathbf{u}$ is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ then the direction cosines are the components of $\frac{\mathbf{u}}{\|\mathbf{u}\|}$

Proof: Text, class

Example: Class

Projections: If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^n$, the vector $\mathbf{v}$ can be decomposed uniquely as the sum of two orthogonal vectors,

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} \text{ (the projection of } \mathbf{v} \text{ onto } \mathbf{u}\text{)}$$
$$\operatorname{proj}_{\mathbf{u}_\perp} \mathbf{v} \text{ (the projection of } \mathbf{v} \text{ orthogonal to } \mathbf{u}\text{)}$$

where $\operatorname{proj}_{\mathbf{u}} \mathbf{v} \parallel \mathbf{u}$ (ie, has the same direction as $\mathbf{u}$)

Theorem: If If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^n$, then

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = (\operatorname{comp}_{\mathbf{u}} \mathbf{v})\frac{\mathbf{u}}{\|\mathbf{u}\|} = (\frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2})\mathbf{u}$$
$$\operatorname{proj}_{\mathbf{u}_\perp} \mathbf{v} = \mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}$$

where $\operatorname{comp}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|}$ is the length of the projection of $\mathbf{v}$ onto $\mathbf{u}$

Derivation and examples: class